Mesoscopic fast-reaction approach of triangular cross-diffusion systems and application to population dynamics

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We consider the following class of triangular cross-diffusion system, on a bounded domain $\Omega \subset \mathrm{R}^N$,

$$\begin{cases} \partial_t u = \Delta\big[(d_u + \varphi(u,v))u\big] + f(u,v), & (0,\infty)\times\Omega, \\ \partial_t v = d_v \Delta v + g(u,v), & (0,\infty)\times\Omega, \end{cases} \tag{1}$$

endowed with homogeneous Neumann boundary conditions and positive initial data. The unknowns u and v are the densities of two species U and V in competition and the reaction terms f and g are chosen accordingly (for instance as Lotka-Volterra competition type reaction functions).
Systems (1) are rich mathematical objects of great complexity with relevant applications in population dynamics. This poster aims to show an approximation procedure of (1) through a system with linear diffusion and additional fast reaction terms. The approximating model describes the evolution of sub-populations of the species U or V, at the mesoscopic scale. Classical results for linear parabolic systems allow us to obtain the global existence of classical solutions of the mesoscopic system. Moreover, we use entropy methods to obtain (1) as the fast reaction limit, together with the existence of weak solutions [2, 3, 4].
As an application, we consider the case where one species (say U) has a more diverse diet than the other. The approximating linear system is then

$$\begin{cases} \partial_t u_a^\varepsilon = d_a \Delta u_a^\varepsilon + f_a(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon) + \frac{1}{\varepsilon} Q(u_a^\varepsilon,\, u_b^\varepsilon,\, v^\varepsilon\,), & (0,+\infty)\times\Omega, \\ \partial_t u_b^\varepsilon = d_b \Delta u_b^\varepsilon + f_b(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon) - \frac{1}{\varepsilon} Q(u_a^\varepsilon,\, u_b^\varepsilon,\, v^\varepsilon\,), & (0,+\infty)\times\Omega, \\ \partial_t v^\varepsilon = d_v \Delta v^\varepsilon + f_v(u_a^\varepsilon, u_b^\varepsilon, v^\varepsilon), & (0,+\infty)\times\Omega, \end{cases} \tag{2}$$

where $Q(u_a, u_b, v) = \phi\Big(\frac{u_b + v}{b}\Big)\, u_b - \psi\Big(\frac{u_a}{a}\Big)\, u_a$. Under suitable assumptions on ϕ, ψ and the initial data, we are able to prove that, as $\epsilon \to 0$, the classical solution of (2) converges towards a weak solution of a cross diffusion system of type (1), with $u = \lim_{\varepsilon\to 0}(u_a^\varepsilon + u_b^\varepsilon)$ and $v = \lim_{\varepsilon\to 0} v^\varepsilon$, [1].
The linear stability of homogeneous equilibria of those systems will be also pointed out and numerical simulations will be included to illustrate the theoretical results.
These results are joint works with Lucilla Corrias (LaMME, Université d'Evry-Paris Saclay), Laurent Desvillettes (IMJ-PRG, Paris, France), Helge Dietert (IMJ-PRG, Paris, France) and Yong-Jung Kim (KAIST, Daejeon, Korea).

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