Quadratic behaviors of 1D linear Schrödinger equation, with bilinear control

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Given a system, whose evolution is given by an ODE or a PDE, upon which we can act through a control, one can take an interest in the question of controllability : can we choose the control such that, the solution, starting from a chosen initial data, reached, at a given time, a given target ? For example, the behaviour of a quantum particle stuck inside an infinite square potential, and subjected to an uniform electric field whose amplitude can be given by $u(t)$, can be described by the Schrödinger equation :

$$\begin{cases} i\partial_t \psi(t,x) = -\partial_x^2 \psi(t,x) - u(t)\mu(x)\psi(t,x), & (t,x) \in (0,T) \times (0,1), \\ \psi(t,0) = \psi(t,1) = 0. \end{cases} \tag{1}$$

This equation is a bilinear control system where,

— the state is the wave function ψ,

— $u : (0,T) \to \mathbb{R}$ denotes a scalar control,

— and $\mu : (0,1) \to \mathbb{R}$ depicts the dipolar moment of the quantum particle.

One can take an interest in the notion of small time controllability around the ground state : for any time $T > 0$, any initial data ψ_0 and any target ψ_f close enough to the ground state (in a sense to specify), can we find a control u such that the solution ψ of the Schrödinger equation (1) with initial condition ψ_0 reaches the target ψ_f at time T, that is $\psi(T) = \psi_f$?

Often, to prove such result, one can start by looking at the controllability of the linearized equation. If the linearized system is indeed controllable, using an inverse mapping theorem, one can hope to prove small time controllability of the nonlinear system. This already has been done for the Schrödinger equation (1) by Beauchard and Laurent in [1] under the assumption on the dipolar moment $\mu \in H^3(0,1)$ that there exists a constant $c > 0$ such that

$$\left| \int_0^1 \mu(x) \sin(\pi x) \sin(j\pi x) dx \right| \geq \frac{c}{j^3}, \quad \forall j \in \mathbb{N}^*. \tag{2}$$

But here, we are looking at situations where we loose the controllability of the linearized system, meaning assumption (2) doesn't hold anymore. In [4], Beauchard and Morancey proved, under the assumption on μ that $\int_0^1 \mu(x)\sin(\pi x)\sin(K\pi x)dx = 0$ for some $K \in \mathbb{N}^*$, doing an power serie expansion of the solution at order two, that the quadratic term induces a drift in the solution, quantified by the norm H^{-1}-norm of the control, denying controllability. In light of the results of Beauchard and Marbach, in finite dimension in [2] and on a parabolic equation in [3], we generalized such result and formulate, for any integer k, assumptions on the dipolar moment under which one may observe a quadratic drift of the solution, quantified by the norm H^{-k}-norm of the control, denying controllability.

Références

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