BAMPHI: matrix and transpose-free algorithm for computing the action of combinations of φ-functions in exponential integrators

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The time integration of stiff systems of differential equations as

$$u'(t) = F(t, u(t)), \quad u(0) = u_0,$$

constitutes a heated topic in numerical analysis. In particular, exponential integrators drew a great deal of attention. In fact, similarly to implicit methods, these methods show good stability properties, allowing integration with large time steps [4]. Each exponential integration step of length τ (out of hundreds, thousands or even millions steps) consists of the same operation : let $\mathbf{A}$ be the linear part of $F(t, u(t))$ (or, say, its Jacobian), one shall compute

$$u^{n+1} := \varphi_0(\theta_0\tau\mathbf{A})v_0 + \varphi_1(\theta_1\tau\mathbf{A})v_1 + \ldots + \varphi_p(\theta_p\tau\mathbf{A})v_p,$$

where $\theta_0, \theta_1, \ldots, \theta_p$ are fixed scalars,

$$\varphi_\ell(x) := \sum_{j=0}^{\infty} \frac{x^j}{(j+\ell)!}, \quad \ell = 0, 1, \ldots, p$$

and the vectors $v_0, v_1, \ldots, v_p \in \mathbb{C}^N$ are obtained, in a recursive fashion, as functions of linear combinations of φ–functions applied to vectors connected to the current state of the system u^n.
The authors exploited this peculiarity of exponential integrators and recent advancements in numerical analysis (such as the works [1, 2]) to build a routine for computing the action of the matrix φ–functions arising in the exponential integration steps, called `bamphi`, able to recycle the information gathered through the exponential integration steps and to reach unprecedented levels of speed and accuracy.
In this presentation, we outline some of `bamphi`'s main features and ideas. Then, we show comparisons with other state-of-the-arts routines (such as `kiops` from [3]) on some model PDEs using Runge–Kutta and Rosenbrock exponential integrators, splitting methods [5], and the new low regularity exponential-type integrators developed in [6].

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