Contraction rates for the Vlasov-Fokker-Planck equation and uniform in time propagation of chaos through coupling

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We adapt in a non-linear case a method used by A. Eberle, A. Guillin and R. Zimmer [2] to prove the existence of a contraction rate for the non linear stochastic differential equation of *McKean-Vlasov* type

$$\begin{cases} dX_t = V_t dt \\ dV_t = \sqrt{2} dB_t - V_t dt - \nabla U(X_t)\, dt - \nabla W * \mu_t(X_t)\, dt \\ \mu_t = \text{Law}(X_t)\,. \end{cases} \tag{1}$$

Here, $(X_t, V_t) \in \mathbb{R}^d \times \mathbb{R}^d$, $(B_t)_{t\geq 0}$ is a Brownian motion in dimension d on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and μ_t is the law of the position X_t. The symbol ∇ refers to the gradient operator, and the symbol $*$ to the operation of convolution.
We assume the interaction ∇W is Lipschitz continuous and the confining force ∇U is both Lipschitz continuous (or locally Lipschitz continuous) and greater than a quadratic function. We use coupling methods suggested by A. Eberle [1] to obtain a contraction rate for a Wasserstein distance.
Finally, using the same method, we prove a result of uniform in time propagation of chaos for the related N particles system in $\mathbb{R}^d$ in mean field interaction

$$\forall i \in \{1, .., N\}, \qquad \begin{cases} dX_t^i &= V_t^i dt, \\ dV_t^i &= \sqrt{2} dB_t^i - V_t^i dt - \nabla U\left(X_t^i\right) dt - \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \nabla W\left(X_t^i - X_t^j\right) dt, \end{cases} \tag{2}$$

where X_t^i and V_t^i are respectively the position and the velocity of the i-th particle, and $\left(B_t^i, 1 \leq i \leq N\right)$ are independent Brownian motions in dimension d.

[1] A. Eberle. *Reflection couplings and contraction rates for diffusions.* Probab. Theory Relat. Fields, **166(3-4)**, 851–886, 2016.

[2] A. Eberle, A. Guillin, R. Zimmer. *Couplings and quantitative contraction rates for Langevin dynamics.* Ann. Probab., **47(4)**, 1982–2010, 2019.

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