Influence of sampling on the convergence rates of greedy algorithms for parameter-dependent random variables

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In this talk we will present a mathematical study [4] of the algorithm proposed in [3] where the authors proposed a variance reduction technique for the computation of parameter-dependent expectations using a reduced basis paradigm. A study of the effect of Monte-Carlo sampling on the theoretical properties of greedy algorithms is given. In particular, using some concentration inequalities for the empirical measure in Wasserstein distance proved in [1], sufficient conditions are proved on the number of samples used for the computation of empirical variances at each iteration of the greedy procedure to guarantee that the resulting method algorithm is a weak greedy [2] algorithm with high probability. These theoretical results are not fully practical and therefore an heuristic procedure is proposed to choose the number of Monte-Carlo samples at each iteration, inspired by this theoretical study, which provides satisfactory results on several numerical test cases. Let us give some more details :
Let $p, d \in \mathbb{N}^*$, $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, Z a $\mathbb{R}^d$-valued random vector with probability law ν, $\mathcal{P} \subset \mathbb{R}^p$ a set of parameter values. For all $\mu \in \mathcal{P}$, let $f_\mu : \mathbb{R}^d \to \mathbb{R}$ a real-valued function such that $f_\mu \in L^2_\nu(\mathbb{R}^d)$.

Goal : For all $\mu \in \mathcal{P}$, quickly compute $\mathbb{E}(f_\mu(Z))$.

A control variate using a "Reduced Basis" paradigm : Let $M_{\text{large}}, M_{\text{small}} \in \mathbb{N}^*$ such that $M_{\text{large}} \gg M_{\text{small}}$. Let $(Z_k^{M_{large}})_{1\leq k\leq M_{\text{large}}}$ and $(Z_k^{M_{\text{small}}})_{1\leq k\leq M_{\text{small}}}$ be iid random vectors with law ν.
Assume that we have selected N values of parameters $(\mu_1, \mu_2, .., \mu_N) \in \mathcal{P}^N$ (with N small), and that, in an offline stage, the quantities $(\mathbb{E}_{M_{\text{large}}}(f_{\mu_i}) := \frac{1}{M_{large}} \sum_{k=1}^{M_{large}} f_{\mu_i}\left(Z_k^{M_{large}}\right))_{1\leq i\leq N}$ have been precomputed. For all $\mu \in \mathcal{P}$, we can build an estimator of $\mathbb{E}(f_\mu(Z))$ using a control variate $\bar{f}_\mu(Z)$, for some function $\bar{f}_\mu : \mathbb{R}^d \to \mathbb{R}$ such that :

$$\mathbb{E}(f_\mu(Z)) \approx \mathbb{E}_{M_{\text{large}}}(\bar{f}_\mu) + \mathbb{E}_{M_{\text{small}}}(f_\mu - \bar{f}_\mu)$$

The control variate function $\bar{f}_\mu$ is constructed as "the projection" of f_μ on the vector space $\{f_{\mu_1}, .., f_{\mu_N}\}$. Then, the online estimation of $\mathbb{E}(f_\mu(Z))$ can be done in a complexity of order N. The objective of this work is to analyze a greedy procedure to select the parameters μ_i, and to show that the selected parameters are close, in some sense, to the best possible set of parameters, with high probability.

[1] N.Fournier A.Guillin. On the rate of convergence in wasserstein distance of the empirical measure. *Probability Theory and Related Fields*, 162(3-4) :707–738, 2015.

[2] P. Binev, A. Cohen, W. Dahmen, R. DeVore, G. Petrova, and P. Wojtaszczyk. Convergence rates for greedy algorithms in reduced basis methods. *SIAM Journal on Mathematical Analysis*, 43(3) :1457–1472, 2011.

[3] S.Boyaval T.Lelièvre. A variance reduction method for parametrized stochastic differential equations using the reduced basis paradigm. *Communications in Mathematical Sciences*, 8(3) :735–762, 2010.

[4] M.R.Blel V.Ehrlacher, T.Lelièvre. Influence of sampling on the convergence rates of greedy algorithms for parameter-dependent random variables. *In preparation to submission*, 2021.

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