Uniform global asymptotic synchronization of a network of Kuramoto oscillators via hybrid coupling

Simone MARIANO, Université de Lorraine, CNRS, CRAN - Nancy, France
Riccardo BERTOLLO, DII,University of Trento - Trento, Italy
Elena PANTELEY, CNRS, L2S - Gif-sur-Yvette, France
Romain POSTOYAN, Université de Lorraine, CNRS, CRAN - Nancy, France
Luca ZACCARIAN, CNRS, LAAS, Université de Toulouse - Toulouse, France,
& DII,University of Trento - Trento, Italy

Kuramoto oscillators are used in various fields to model and analyze dynamics of a broad family of systems with oscillatory behavior see, e.g. [3, 4]. Among the phenomena characterizing oscillating systems, collective synchronization plays a key role. When the network comprises oscillators with the same natural frequency, it is now well-known that the system admits multiple synchronization sets some of which are unstable see, e.g. [2]. The downside of this result is that the closer a solution is initialized to an unstable set, the longer it will take for phase synchronization to arise: we talk of non-uniform convergence, which can be problematic in engineering applications.
In this context, we propose a new Kuramoto-like model, which exploits the periodicity of phases and which relies on hybrid techniques, in the sense that the obtained system exhibits both continuous-time and discrete-time dynamics [1]. Hence, we propose a novel hybrid model for an n-agent network of identical oscillators interconnected over an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is the set of all the nodes and $\mathcal{E}$ the set of all the edges, with $n = |\mathcal{V}|$ nodes and $m = |\mathcal{E}|$ edges. The continuous dynamics of each oscillator, represented by a node in $\mathcal{G}$, is defined as

$$\dot{\theta}_i = \omega + \gamma \sum_{j \in \mathcal{V}_i} \sigma(\theta_j - \theta_i + 2k_{ij}\pi), \qquad x \in C \tag{1}$$

where $\theta_i \in [-\pi - \delta, \pi + \delta]$, is the phase of oscillator i, $\omega \in \mathbb{R}$ is the natural frequency, which is the same for all oscillators, with $\delta \in (0, \pi)$ and $\gamma \in \mathbb{R}_{>0}$. Variable k_{ij}, $(i, j) \in \mathcal{E}$, is a logic state, taking values in $\{-1, 0, 1\}$. Its role is to unwind the difference between the two phases θ_j and θ_i through jumps while remaining constant along flows. The flow set C is selected as the closed complement of the jump set D where the values of k_{ij} and/or the phase exhibit a jump, which is not described in this abstract. Function σ, whose domain is the interval $[-\pi - \delta, \pi + \delta]$, is a symmetric, continuous function and it is selected such that $\sigma(s) = 0$ if and only if $s = 0$. We prove that the synchronization set, defined as

$$\mathcal{A} := \{x \in C \cup D : \theta_i = \theta_j + 2k_{ij}\pi, \forall (i, j) \in \mathcal{E}\}, \tag{2}$$

is uniformly globally asymptotically stable thereby ensuring uniform convergence as desired when the interconnection graph is a tree. We also show that the model preserves the original behavior of Kuramoto oscillators near the phase synchronization set.
Simulation results are provided to illustrate the theoretical guarantees and demonstrate the potential strength of hybrid theoretical tools to overcome fundamental limitations of continuous-time networked systems.

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Contact : simone.mariano@univ-lorraine.fr