Multi-scale finite element methods for advection-diffusion problems

Rutger BIEZEMANS, CERMICS and MATHERIALS - Champs-sur-Marne

The multi-scale finite element method (MsFEM) is a finite element (FE) approach that allows to solve partial differential equations (PDEs) with highly oscillatory coefficients on a coarse mesh, i.e., a mesh with elements of size much larger than the characteristic scale of the oscillations [2, 3]. To do so, MsFEMs use pre-computed basis functions adapted to the differential operator that take into account the small scales of the problem.

The MsFEM theory is mostly developed for symmetric elliptic problems. Additional difficulties arise in the numerical approximation of PDEs with dominating advection terms. Such problems are challenging even with non-oscillating coefficients. Their solutions typically present boundary layers, characterized by steep gradients ; naive FE approximation may lead to spurious oscillations even outside the boundary layer. Multiple stabilization methods exist today to adequately adapt FE methods to the resolution of advection-diffusion problems [5]. In spite of various proposals to combine multi-scale and stabilization methods, a universally best method has not yet been identified.

In this contribution, we will first recall the main challenges of numerically solving advection-diffusion equations with multi-scale coefficients and next summarize some recent attempts to approximate such PDEs by combining MsFEM and stabilization approaches [4]. In addition, we will present new variants that are currently being investigated [1]. Differences in the performance of the various methods will be illustrated with the help of numerical experiments. We shall also discuss the connections between MsFEM variants and the homogenization theory that we have established for the solutions of these PDEs outside the boundary layers (rigorous derivation in one dimension, and formal derivation in the higher dimensional setting).

The work described in this communication is partly joint work with Alexei Lozinski (Université de Besançon), Frédéric Legoll and Claude Le Bris (Ecole des Ponts and Inria). The support of DIM Math INNOV and Inria is gratefully acknowledged. We also acknowledge the partial support from ONR and EOARD.

[1] R. A. Biezemans. Phd thesis, École Nationale des Ponts et Chaussées. In preparation.

[2] Y. Efendiev, T. Hou. *Multiscale Finite Element Methods.* Springer-Verlag New-York, 2009.

[3] T. Y. Hou, X.-H. Wu. *A multiscale finite element method for elliptic problems in composite materials and porous media.* Journal of Computational Physics, **134(1)**, 169–189, 1997. doi : https ://doi.org/10.1006/jcph.1997.5682.

[4] C. Le Bris, F. Legoll, F. Madiot. *A numerical comparison of some multiscale finite element approaches for advection-dominated problems in heterogeneous media.* ESAIM : M2AN, **51(3)**, 851–888, 2017. doi :10.1051/m2an/2016057.

[5] A. Quarteroni. *Numerical Models for Differential Problems.* Springer-Verlag Mailand, second ed., 2014.

Contact : `rutger.biezemans@enpc.fr`