A posteriori error estimates for the time dependent convection-diffusion-reaction equation coupled with the Navier-Stokes system

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Plan

Convection-diffusion-reaction equation coupled with the Navier-Stokes system

Ω : a connected bounded open domain in $\mathbb{R}^d$ ($d = 2,\ 3$), $\partial\Omega$: boundary of Ω.

$$\left\{\begin{array}{rcll}
\dfrac{\partial \mathbf{u}}{\partial t}(x,t) - \mathrm{div}(\nu(C(x,t))\nabla \mathbf{u}(x.t)) & & & \\
+(\mathbf{u}(x,t)\cdot\nabla)\mathbf{u}(x,t) + \nabla p(x,t) & = & \mathbf{f}(x,t,C(x,t)) & \text{in } \Omega\times]0,T[, \\
\mathrm{div}\ \mathbf{u}(x,t) & = & 0 & \text{in } \Omega\times]0,T[, \\
\dfrac{\partial C}{\partial t}(x,t) + (\mathbf{u}(x,t)\cdot\nabla)C(x,t) - \alpha\Delta C(x,t) + r_0 C(x,t) & = & g(x,t) & \text{in } \Omega\times]0,T[, \\
\mathbf{u}(x,t) & = & 0 & \text{on } \partial\Omega\times]0,T[, \\
C(x,t) & = & 0 & \text{on } \partial\Omega\times]0,T[, \\
\mathbf{u}(x,0) & = & \mathbf{u}_0 & \text{in } \Omega, \\
C(x,0) & = & C_0 & \text{in } \Omega.
\end{array}\right.$$

- $\mathbf{u}$: fluid velocity.
- p : fluid pressure.
- C : concentration in the fluid.
- ν : fluid viscosity.
- α : diffusion coefficient.
- r_0 : positive constant.
- $\mathbf{f} = (f_1, ..., f_d) \in H^{-1}(\Omega)^d$: external force.
- g : external concentration source.

NB : Our main purpose in this thesis is to study the case where α depends on the concentration C, but as a first step we will consider α constant.

Variational formulation

$X = H_0^1(\Omega)^d,\ M = L_0^2(\Omega)$ and $Y = H_0^1(\Omega)$,

$$
(E)\begin{cases}
\text{Find } (\mathbf{u}, p, C) \in X \times M \times Y \text{ such that} \\
\dfrac{d}{dt}(\mathbf{u}(t), \mathbf{v}) + (\nu(C(t))\nabla \mathbf{u}(t), \nabla \mathbf{v}) - (p(t), \operatorname{div}\ \mathbf{v}) + c_{\mathbf{u}}(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}) = (\mathbf{f}(t, C(t)), \mathbf{v}) & \forall \mathbf{v} \in X, \\
(\operatorname{div}\ \mathbf{u}(t), q) = 0 & \forall q \in M, \\
\dfrac{d}{dt}(C(t), r) + c_C(\mathbf{u}(t), C(t), r) + \alpha(\nabla C(t), \nabla r) + r_0(C(t), r) = (g(t), r) & \forall r \in Y, \\
\mathbf{u}(0) = \mathbf{u}_0 & \text{in } \Omega, \\
C(0) = C_0 & \text{in } \Omega.
\end{cases}
$$

with

$$c_{\mathbf{u}}(\mathbf{u}(t), \mathbf{u}(t), \mathbf{v}) = \int_\Omega ((\mathbf{u}(t) \cdot \nabla)\mathbf{u}(t)) \cdot \mathbf{v}\ dt \qquad \text{and} \qquad c_C(\mathbf{u}(t), C(t), r) = \int_\Omega ((\mathbf{u}(t) \cdot \nabla) C(t))\ r\ dt\,.$$

Hypothesis

Assumption

i)
$$\mathbf{f}(x,t,C(x,t)) = \mathbf{f}_0(x,t) + \mathbf{f}_1(x,C(x,t)),$$
where $\mathbf{f}_0 \in C^0(0,T;L^2(\Omega)^d)$ and $\mathbf{f}_1$ is $c^*_{\mathbf{f}_1}$-lipschitz with respect to its second argument, and
$$\forall x \in \Omega, \forall \xi \in \mathbb{R}, |\mathbf{f}_1(x,\xi)| \leq c_{\mathbf{f}_1}|\xi|,$$
where $c_{\mathbf{f}_1}$ is a positive constant.

ii) $g \in C^0(0,T;L^2(\Omega))$,

iii) $\nu \in L^\infty(\mathbb{R})$ and is Lipschitz-continuous, with Lipschitz constant c_ν. Furthermore, there exist two positive constants $\hat{\nu}_1$ and $\hat{\nu}_2$ such that, for any $\theta \in \mathbb{R}$.
$$\hat{\nu}_1 \leq \nu(\theta) \leq \hat{\nu}_2.$$

iv) $\mathbf{u}_0 \in L^2(\Omega)^d$, $\operatorname{div} \mathbf{u}_0 = 0$, $\mathbf{u}_0 \cdot \mathbf{n}_{\partial\Omega} = 0$ and $C_0 \in L^2(\Omega)$.

Existence and uniqueness

Theorem

Under the fixed Hypothesis, Problem (E) admits at least one solution

$$(\mathbf{u}, p, C) \in L^2(0,T; H_0^1(\Omega)^d) \cap L^\infty(0,T; L^2(\Omega)^d) \times L^2(0,T; L^2(\Omega)) \times L^2(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega)).$$

This solution is unique if

$$\mathbf{u} \in L^p(0,T; W^{1,r}(\Omega)^d), \text{ where } p \geq 4 \text{ and } r \geq 4.$$

Every solution of (E) satisfies the bound

$$\begin{aligned} &\| \mathbf{u} \|_{L^\infty(0,T;L^2(\Omega)^d)} + \| \mathbf{u} \|_{L^2(0,T;H_0^1(\Omega)^d)} + \| C \|_{L^\infty(0,T;L^2(\Omega))} + \| C \|_{L^2(0,T;H_0^1(\Omega))} \\ &\leq \hat{C}\Big(\| g \|_{L^2(0,T;L^2(\Omega))} + \| \mathbf{f}_0 \|_{L^2(0,T;L^2(\Omega)^d)} + \| \mathbf{u}_0 \|_{L^2(\Omega)^d} + \| C_0 \|_{L^2(\Omega)} \Big) \end{aligned}$$

where $\hat{C}$ is a positive constant which depends of $S_2^0, \hat{\nu}_1, \alpha, r_0$ and $c_{\mathbf{f}_1}$.

Aldbaissy R., Hecht F., Sayah T., Mansour G., A full discretisation of the time-dependent Boussinesq (buoyancy) model with nonlinear viscosity. Calcolo, 4 (2018).

Space and time discretization

- We use a regular mesh.
- We descritize $(\mathbf{u}, p, C)$ using finite element $(P_{1b}/P_1/P_1)$.
- We discritize in time using semi-implicit Euler method.

Let X_{nh}, M_{nh} and Y_{nh} such that $X_{nh} \subset X, M_{nh} \subset M$ and $Y_{nh} \subset Y$, and for each $n \in \{1, \cdots, N\}$,

$$
\begin{aligned}
Z_{nh} &= \{q_h \in C^0(\bar{\Omega})\ \forall \kappa \in \mathcal{T}_{nh},\ q_{h|\kappa} \in P_1\},\\
X_{nh} &= \big\{\mathbf{v}_h \in C^0(\bar{\Omega})^d\ ; \forall \kappa \in \mathcal{T}_{nh},\ \mathbf{v}_{h|\kappa} \in P_{1b}, \mathbf{v}_{h|\partial\Omega=0}\big\},\\
Y_{nh} &= \big\{r_h \in Z_{nh};\ r_{h|\partial\Omega=0}\big\},\\
M_{nh} &= \big\{q_h \in Z_{nh}; \int_\Omega q_h\, \mathbf{dx} = 0\big\}.
\end{aligned}
$$

Full discrete scheme

For every $n \in \{1, \cdots, N\}$, having $\mathbf{u}_h^{n-1} \in X_{(n-1)h}$ and $C_h^{n-1} \in Y_{(n-1)h}$, Find $(\mathbf{u}_h^n, p_h^n) \in X_{nh} \times M_{nh}, C_h^n \in Y_{nh}$ such that,

$$
(\text{Eds1}) \left\{ \begin{array}{ll}
\dfrac{1}{\tau_n}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{v}_h) + (\nu(C_h^{n-1})\nabla \mathbf{u}_h^n, \nabla \mathbf{v}_h) & \\
\qquad + d_{\mathbf{u}}(\mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \mathbf{v}_h) - (p_h^n, \text{div }\ \mathbf{v}_h) = (\mathbf{f}^n(C_h^{n-1}), \mathbf{v}_h) & \forall \mathbf{v}_h \in X_{nh}, \\
& \\
(q_h, \text{div }\ \mathbf{u}_h^n) = 0 & \forall q_h \in M_{nh}, \\
& \\
\dfrac{1}{\tau_n}(C_h^n - C_h^{n-1}, r_h) + d_C(\mathbf{u}_h^n, C_h^n, r_h) & \\
\qquad + \alpha(\nabla C_h^n, \nabla r_h) + r_0(C_h^n, r_h) = (g^n, r_h) & \forall r_h \in Y_{nh},
\end{array} \right.
$$

with

$$
d_{\mathbf{u}}(\mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \mathbf{v}_h) = ((\mathbf{u}_h^{n-1} \cdot \nabla)\mathbf{u}_h^n, \mathbf{v}_h) + \frac{1}{2}(\text{div}(\mathbf{u}_h^{n-1})\mathbf{u}_h^n, \mathbf{v}_h)
$$

and

$$
d_C(\mathbf{u}_h^n, C_h^n, r_h) = ((\mathbf{u}_h^n \cdot \nabla)C_h^n, r_h) + \frac{1}{2}(\text{div}(\mathbf{u}_h^n)C_h^n, r_h).
$$

Existence and uniqueness

Theorem

At each time step n, for a given $\mathbf{u}_h^{n-1} \in X_{(n-1)h}, C_h^{n-1} \in Y_{(n-1)h}$ and under the fixed Assumption, problem (Eds1) admits a unique solution $(\mathbf{u}_h^n, p_h^n, C_h^n) \in X_{nh} \times M_{nh} \times Y_{nh}$. Furthermore we have, for $m = 1, \cdots, N$, the following bounds

$$\frac{1}{2} \parallel \mathbf{u}_h^m \parallel^2_{L^2(\Omega)^d} + \frac{\hat{\nu}_1}{2} \sum_{n=0}^{m} \tau_n |\mathbf{u}_h^n|^2_{H_0^1(\Omega)^d} \\ \leq \tilde{C}_d \Big(\sum_{n=0}^{m} \tau_n \parallel g^n \parallel^2_{L^2(\Omega)} + \sum_{n=0}^{m} \tau_n \parallel \mathbf{f}_0^n \parallel^2_{L^2(\Omega)} + \parallel C_h^0 \parallel^2_{L^2(\Omega)} + \parallel \mathbf{u}_h^0 \parallel^2_{L^2(\Omega)^d} \Big),$$

$$\frac{1}{2} \parallel C_h^m \parallel^2_{L^2(\Omega)} + \frac{\alpha}{2} \sum_{n=0}^{m} \tau_n |C_h^n|^2_{H_0^1(\Omega)} + r_0 \sum_{n=0}^{m} \tau_n \parallel C_h^n \parallel^2_{L^2(\Omega)} \\ \leq \tilde{C'}_d \Big(\sum_{n=0}^{m} \tau_n \parallel g^n \parallel^2_{L^2(\Omega)} + \parallel C_h^0 \parallel^2_{L^2(\Omega)} \Big).$$

where $\tilde{C}_d$ et $\tilde{C'}_d$ are positive constants independent of h and m.

A posteriori error

- Putting up the local error indicators in time and in space
- Bounding the solution error :

$$\begin{array}{c} C_1 \text{ indicators} \\ + \text{ data oscillations} \end{array} \leq ||u - u_h|| \leq \begin{array}{c} C_2 \text{ indicators} \\ + \text{ data oscillations} \end{array}$$

- Mesh refinement :

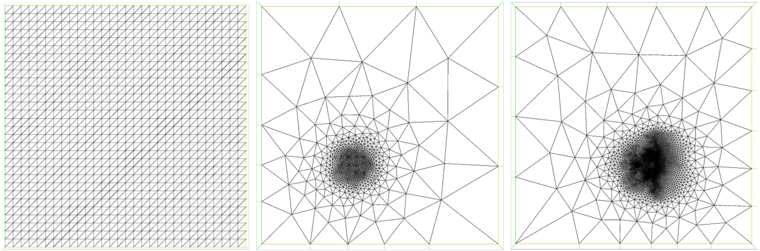

FIGURE – Mesh evolution

Indicators

$$(\eta^{\tau}_{u,n,\kappa_n})^2 = \tau_n \parallel \mathbf{u}_h^n - \mathbf{u}_h^{n-1} \parallel^2_{H^1(\kappa_n)},$$

$$(\eta^{\tau}_{c,n,\kappa_n})^2 = \tau_n \parallel C_h^n - C_h^{n-1} \parallel^2_{H^1(\kappa_n)},$$

$$\begin{aligned}(\eta^{h}_{u,n,\kappa_n})^2 = \quad & h^2_{\kappa_n} \parallel \mathbf{f}^n(C_h^{n-1}) - \frac{1}{\tau_n}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}) + \operatorname{div}(\nu(C_h^{n-1})\nabla \mathbf{u}_h^n) - \mathbf{u}_h^{n-1}\cdot\nabla\mathbf{u}_h^n \\ & -\frac{1}{2}\operatorname{div}\mathbf{u}_h^{n-1}\mathbf{u}_h^n - \nabla p_h^n)(x) \parallel^2_{0,\kappa_n} + \frac{1}{2}\sum_{e_n\in\varepsilon_{\kappa n}} h_{e_n} \parallel [(\nu(C_h^{n-1})\nabla\mathbf{u}_h^n - p_h^n\mathbb{I})(\sigma)]_{e_n}\mathbf{n} \parallel^2_{0,e_n} \\ & + \parallel \operatorname{div}\mathbf{u}_h^n \parallel^2_{0,\kappa_n},\\ (\eta^{h}_{c,n,\kappa_n})^2 = \quad & h^2_{\kappa_n} \parallel g^n - \frac{1}{\tau_n}(C_h^n - C_h^{n-1}) + \alpha\Delta C_h^n - \mathbf{u}_h^n\cdot\nabla C_h^n - \frac{1}{2}\operatorname{div}(\mathbf{u}_h^n)C_h^n - r_0 C_h^n \parallel^2_{0,\kappa_n} \\ & + \frac{1}{2}\sum_{e_n\in\varepsilon_{\kappa n}} h_{e_n} \parallel [\alpha\nabla C_h^n(\sigma)]_{e_n}\cdot\mathbf{n} \parallel^2_{0,e_n}.\end{aligned}$$

We prove optimal *a posteriori* error estimates by using the norms

$$[[\mathbf{u}-\mathbf{u}_h]] = \Big(\| \mathbf{u}(t_n) - \mathbf{u}_h(t_n) \|^2_{L^2(\Omega)^2} + \hat{\nu}_1 \ \max \Big(\int_0^{t_n} \| \mathbf{u}(t) - \mathbf{u}_h(t) \|^2_X \, dt, \sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} \| \mathbf{u}(t) - \pi_\tau \mathbf{u}_h(t) \|^2_X \, dt \Big) \Big)^{1/2}$$

and

$$[[C-C_h]] = \Big(\| C(t_n) - C_h(t_n) \|^2_{L^2(\Omega)^2} + \alpha \ \max \Big(\int_0^{t_n} \| C(t) - C_h(t) \|^2_Y \, dt, \sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} \| C(t) - \pi_\tau C_h(t) \|^2_Y \, dt \Big) \Big)^{1/2}.$$

Upper bound

Theorem

Let $h_n \leq c_s \tau_n, \forall n \in \{1, \ldots, N\}$, where c_s is a positive constant independent of n. for each $m \in \{1, \cdots, N\}$, the solutions $(\mathbf{u}, p, C)$ of (E) and $(\mathbf{u}_h, p_h, C_h)$ of (Eds1) satisfy the following *a posteriori* estimation :

$$
\begin{aligned}
&[[\mathbf{u}-\mathbf{u}_h]]^2(t_m) + \| \frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h) + \mathbf{u}\cdot\nabla\mathbf{u} - \pi_{l,\tau}\mathbf{u}_h \cdot \nabla \pi_\tau \mathbf{u}_h - \frac{1}{2}\,\mathrm{div}(\pi_{l,\tau}\mathbf{u}_h)\pi_\tau\mathbf{u}_h + \nabla(p-p_h) \|_{L^2(0,t_m;X')} \\
&\quad + [[C-C_h]]^2(t_m) + \| \frac{\partial}{\partial t}(C-C_h) + \mathbf{u}\cdot\nabla C - \pi_\tau\mathbf{u}_h\cdot\nabla\pi_\tau C_h - \frac{1}{2}\,\mathrm{div}(\pi_\tau\mathbf{u}_h)\pi_\tau C_h \|_{L^2(0,t_m;Y')} \\
&\leq c\Big(\| g-\pi_\tau g \|^2_{L^2(0,t_m;Y')} + \| C_0 - C_h^0 \|^2_{L^2(\Omega)} + \| \mathbf{u}_0 - \mathbf{u}_h^0 \|^2_{L^2(\Omega)} + \tau_0 \| \mathbf{u}_0 - \mathbf{u}_h^0 \|^2_X \\
&\quad + \| \mathbf{f}_0 - \pi_\tau \mathbf{f}_0 \|^2_{L^2(0,t_m;L^2(\Omega)^2)} + \sum_{n=1}^{m} \sum_{\kappa_n \in \tau_{nh}} \big((\eta^\tau_{c,n,\kappa_n})^2 + (\eta^\tau_{u,n,\kappa_n})^2 + \tau_n (\eta^h_{c,n,\kappa_n})^2 \big) + \tau_n (\eta^h_{u,n,\kappa_n})^2 \big) \Big).
\end{aligned}
$$

Lower bound

Theorem

We have for all $n \in \{1, \cdots, N\}$:

- $(\eta^{\tau}_{c,n,\kappa_n})^2 \leq c\ \big(\ \| \ C - C_h \ \|^2_{L^2(t_{n-1},t_n;H^1(\kappa_n))} + \| \ C - \pi_\tau C_h \ \|^2_{L^2(t_{n-1},t_n;H^1(\kappa_n))}\ \big)$,

- $\tau_n(\eta^h_{c,n,\kappa_n})^2 \leq$
$$c\Big(\ \| \ \frac{\partial}{\partial t}(C - C_h)(t) + \mathbf{u}\cdot\nabla C - \mathbf{u}^n_h\cdot\nabla C^n_h - \frac{1}{2}\,\mathrm{div}\,\mathbf{u}^n_h C^n_h + r_0(C - C^n_h) \ \|^2_{L^2(t_{n-1},t_n;H^{-1}(\kappa_n))} \Big)$$
$$+ \| \ C - \pi_\tau C_h \ \|^2_{L^2(t_{n-1},t_n;H^1(\kappa_n))} + \| \ g - \pi_\tau g \ \|^2_{L^2(t_{n-1},t_n;H^{-1}(\kappa_n))} + \tau_n h^2_{\kappa_n} \ \| \ g^n - g^n_h \ \|^2_{L^2(\kappa_n)} \Big).$$

Lower bound

Theorem

Let $\nabla \mathbf{u} \in L^\infty(0,T;L^4(\Omega)^{2\times 2})$, we have for all $n \in \{1,\cdots,N\}$:

- $(\eta^{\tau}_{u,n,\kappa_n})^2 \leq c\ \big(\ \| \mathbf{u} - \mathbf{u}_h \|^2_{L^2(t_{n-1},t_n;H^1(\kappa_n)^2)} + \| \mathbf{u} - \pi_\tau \mathbf{u}_h \|^2_{L^2(t_{n-1},t_n;H^1(\kappa_n)^2)}\ \big)$,

- $$\begin{aligned}\tau_n(\eta^h_{u,n,\kappa_n})^2 \leq\ & c\Big(\| \frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h) + (\mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{u}_h^{n-1}\cdot\nabla\mathbf{u}_h^n - \frac{1}{2}\operatorname{div}(\mathbf{u}_h^{n-1})\mathbf{u}_h^n + \nabla(p-p_h) \|_{L^2(t_{n-1},t_n;H^{-1}(\kappa_n)^2)} \\ & + \| C - C_h \|_{L^2(t_{n-1},t_n;L^2(\kappa_n))} + \quad \tau_n \| C_h^n - C_h^{n-1} \|_{L^2(\kappa_n)} + \| \mathbf{u} - \mathbf{u}_h \|_{L^2(t_{n-1},t_n;H^1(\kappa_n))} \\ & + \| \mathbf{u}_h^n - \mathbf{u} \|_{L^2(t_{n-1},t_n;H^1(\kappa_n))} + \| \nu_h(C) - \nu(C) \|_{L^2(t_{n-1},t_n;L^2(\kappa_n))} \\ & + \| \mathbf{f}_0 - \mathbf{f}_0(t_n) \|_{L^2(t_{n-1},t_n;L^2(\kappa_n))} + h^2_{\kappa_n} \| \mathbf{f}_h^n(C) - \mathbf{f}^n(C) \|_{L^2(t_{n-1},t_n;H^{-1}(\kappa_n))^2} \Big).\end{aligned}$$

Numerical Results

- FreeFem++.
- $\Omega = [0,1]^2$, $T = 1$, $\alpha = 1$, $r_0 = 1$ and $\nu(C) = 0.2\sin(C) + 1$.
- The exact solution $(\mathbf{u}_{ex}, p_{ex}, C_{ex}) = (\mathbf{rot}\ \psi, p_{ex}, C_{ex})$, where

$$\psi(x,y,t) = x^2(x-1)^2\,y^2(y-1)^2\ \sin(t),$$
$$p_{ex}(x,y,t) = (t+1)\ \cos(\pi x)\ \cos(\pi y),$$
$$C_{ex}(x,y,t) = -t\,e^{-100((x-0.3-0.3t)^2+(y-0.3)^2)}.$$

- The initial time step $\tau = \dfrac{1}{N}$ and an initial mesh corresponding to $N = 20$.

Algorithm :

(1) $\mathbf{u}_h^n \rightarrow \mathbf{u}_h^{n+1}$ and $C_h^n \rightarrow C_h^{n+1}$.
We calculate the indicators.

(2) If the error is less than a fixed tolerance $\rightarrow$ go to the next time step $\rightarrow$ go to **(1)**.

Other case $\rightarrow$ **(3)**.

(3) if the error in space is smaller than the error in time $\rightarrow$ adaptation of the time step $\rightarrow$ go to **(1)** .

if the error in time is smaller than the error in space $\rightarrow$ adaptation of the mesh $\rightarrow$ go to **(1)** .

Mesh evolution

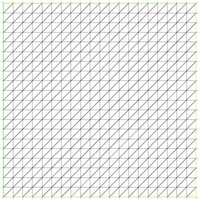

FIGURE – Initial mesh

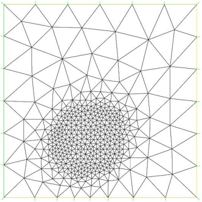

FIGURE – Mesh at $t = 0.496$

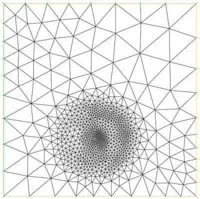

FIGURE – Mesh at $t = 0.768$

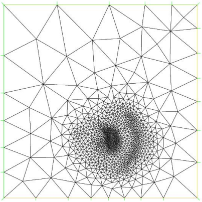

FIGURE – Mesh at $t = 0.897$

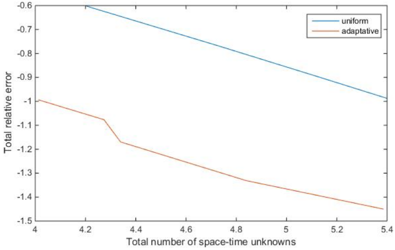

FIGURE – Total relative errors .

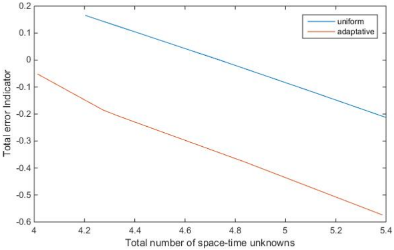

FIGURE – Total indicators erreur .

- Total error reduce by $26,59\%$
- Error indicator reduced by 48.66%
- Time reduced by $40\ \%$

We define the efficiency index as following :

$$EI = \left(\frac{\displaystyle\sum_{n=1}^{N} \sum_{\kappa_n \in \mathcal{T}_{nh}} (\tau_n (\eta^{\tau}_{u,n,\kappa_n})^2 + \tau_n (\eta^{\tau}_{c,n,\kappa_n})^2 + \tau_n (\eta^{h}_{u,n,\kappa_n})^2 + \tau_n (\eta^{h}_{c,n,\kappa_n})^2}{\displaystyle\sum_{n=1}^{N} \tau_n \big(|\mathbf{u}_h^n - \mathbf{u}^n|^2_{H^1(\Omega)^2} + ||p_h^n - p^n||^2_{L^2(\Omega)} + |C_h^n - C^n|^2_{H^1(\Omega)} \big)} \right)^{1/2}.$$

STU	10 308	18 767	21 857	69 065	242 458
EI	8.75	7.79	9.12	8.97	7.52

TABLE – Efficiency index with respect to the total number of space-time unknowns.

Conclusion

- Convection-diffusion-reaction equation coupled with the Navier-Stokes system.
- *A posteriori* analysis .
- Validation of theoretical results by numerical simulations.

Perspectives

- Non-constant diffusion term.

References

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THANK YOU FOR YOUR ATTENTION

Proof

Concentration error :

- Upper bound of the error between the solution C of problem (E) and the discrete solution C_h.

The result depends on the velocity error.

Velocity error :

- Auxiliary problem : time-discretization only.
- Upper bound of the error between the exact solution $\mathbf{u}$ of Problem (E) and the time-affine function $\mathbf{u}_\tau$.
- Upper bound of the error between $\mathbf{u}_\tau$ and the discrete solution $\mathbf{u}_h$.
- Combination of these two previous.

The result depends on the concentration error.

The combination of the concentration error and the velocity error gives us the result using the Gronwall Lemma.

Proof

1 - **Concentration error**

Theorem

Let $\mathbf{u}$ and C be the velocity and concentration solutions of problem (E). Supposing that $\mathbf{u} \in L^\infty(0,T;L^3(\Omega)^2)$, $C \in L^\infty(0,T;L^3(\Omega))$ and $\nabla C \in L^\infty(0,T;L^2(\Omega)^2)$, the following *a posteriori error estimate holds between* C *and the solution* C_h *associated to* $(C_h^n)_{0\le n\le N}$*, solution of problem (Eds1) : for* $1 \le m \le N$*,*

$$\begin{aligned}\| C(t_m) - C_h(t_m) \|^2_{L^2(\Omega)} &+ \alpha \int_0^{t_m} \| C(s) - C_h(s) \|^2_Y \, ds + 2r_0 \int_0^{t_m} \| C(s) - C_h(s) \|^2_{L^2(\Omega)} \, ds \\ &\le c \Big(\sum_{n=1}^m \| g - g^n \|^2_{L^2(t_{n-1},t_n;Y')} + \| C_0 - C_h^0 \|^2_{L^2(\Omega)} + \| \mathbf{u} - \mathbf{u}_h \|^2_{L^2(0,t_m;X)} \\ &\quad + \sum_{n=1}^m \sum_{\kappa_n \in \tau_{nh}} \big((\eta^\tau_{c,n,\kappa_n})^2 + (\eta^\tau_{u,n,\kappa_n})^2 + \tau_n (\eta^h_{c,n,\kappa_n})^2 \big) \Big),\end{aligned}$$

where c *is a constant independent of the time and mesh steps.*

Proof

2 - **Velocity error**

We introduce the following time semi-discrete problem :

$$(P_{aux})\begin{cases} \text{Let } C_h^{n-1} \text{ be the concentration component of the finite element solution of (Eds1), then} \\ \text{knowing } \mathbf{u}^{n-1}, \text{ find}(\mathbf{u}^n, p^n) \in X \times M \text{ such that} \\ \\ \dfrac{1}{\tau_n}(\mathbf{u}^n - \mathbf{u}^{n-1}, \mathbf{v}) + (\nu(C_h^{n-1})\nabla \mathbf{u}^n, \nabla \mathbf{v}) \\ \qquad + (\mathbf{u}^{n-1} \cdot \nabla \mathbf{u}^n, \mathbf{v}) - (\mathrm{div}\ \mathbf{v}, p^n) = (\mathbf{f}^n(C_h^{n-1}), \mathbf{v}) \qquad \forall \mathbf{v} \in X, \\ \\ (\mathrm{div}\ \mathbf{u}^n, q) = 0 \qquad \forall q \in M, \end{cases}$$

with $\mathbf{u}^0 = \mathbf{u}_0$ and $\mathbf{f}^n(C_h^{n-1}) = \mathbf{f}_0^n + \mathbf{f}_1(C_h^{n-1})$ with $\mathbf{f}_0^n = \mathbf{f}_0(t_n)$.

Proof

2 - **Velocity error**

Theorem

Let $\mathbf{u}$ be the velocity of problem (3) and $\mathbf{u}_\tau$ the velocity associated to $(\mathbf{u}^n)_{0\leq n\leq N}$ solution of the auxiliary problem (P_{aux}). Suppose that $\nabla\mathbf{u}\in L^\infty(0,T;L^4(\Omega)^{2\times 2})$ and $\mathbf{u}\in L^\infty(0,T;L^3(\Omega)^2)$. Then, the following *a posteriori error estimate holds*

$$\|\mathbf{u}(t)-\mathbf{u}_\tau(t)\|^2_{L^2(\Omega)^2}+\hat{\nu}_1\int_0^t\|\mathbf{u}(\tau)-\mathbf{u}_\tau(\tau)\|^2_X\,d\tau$$
$$\leq\ c\Big(\|\mathbf{f}_0-\pi_\tau\mathbf{f}_0\|^2_{L^2(0,t;L^2(\Omega)^2)}+\|C-\pi_{l,\tau}C_h\|^2_{L^2(0,t;L^4(\Omega))}+\|\pi_\tau\mathbf{u}-\pi_{l,\tau}\mathbf{u}\|^2_{L^2(0,t;X)}\Big).$$

where c is a positive constant independent of the time and mesh steps.

Theorem

Let $t\in]t_{m-1},t_m]$. Let $h_n\leq c_s\tau_n, \forall n\in\{1,\ldots,N\}$, where c_s is a positive constant independent of n. The following *a posteriori* error holds between $\mathbf{u}_\tau(t)$ and $\mathbf{u}_h(t)$:

$$\|(\mathbf{u}_\tau-\mathbf{u}_h)(t)\|^2_{L^2(\Omega)^2}+\int_0^t\|\mathbf{u}_\tau-\mathbf{u}_h\|^2_X(s)ds$$
$$\leq c\Big(\|\mathbf{u}_0-\mathbf{u}_h^0\|^2_{L^2(\Omega)^2}+\tau_0\|\mathbf{u}_0-\mathbf{u}_h^0\|^2_X+\sum_{n=1}^m\sum_{\kappa_n\in\mathcal{T}_{nh}}\tau_n(\eta^h_{u,n,\kappa_n})^2\Big),$$

where c is a positive constant independent of the time and mesh steps.

Proof

2 - **Velocity error**

Theorem

Let $h_n \leq c_s \tau_n, \forall n \in \{1, \dots, N\}$, where c_s is a positive constant independent of n. Under our assumptions of Theorem , for $t \in]t_{m-1}, t_m]$, we have the following *a posteriori* estimation between the velocity $\mathbf{u}$ solution of problem (E) and $\mathbf{u}_h$ corresponding to $\mathbf{u}_h^n$ of problem (Eds1) :

$$\| \mathbf{u}(t) - \mathbf{u}_h(t) \|^2_{L^2(\Omega)^2} + \hat{\nu}_1 \int_0^t \| \mathbf{u}(s) - \mathbf{u}_h(s) \|^2_X \, ds \leq c \Big(\| \mathbf{f}_0 - \pi_\tau \mathbf{f}_0 \|^2_{L^2(0,t;L^2(\Omega)^2)} + \| \mathbf{u}_0 - \mathbf{u}_h^0 \|^2_{L^2(\Omega)^2}$$

$$+ \tau_0 \| \mathbf{u}_0 - \mathbf{u}_h^0 \|^2_X + \| C - \pi_{l,\tau} C_h \|^2_{L^2(0,t,L^4(\Omega))} + \| \pi_\tau \mathbf{u} - \pi_{l,\tau} \mathbf{u} \|^2_{L^2(0,t;X)} + \sum_{n=1}^{m} \tau_n \sum_{\kappa_n \in \mathcal{T}_{nh}} (\eta^h_{u,n,\kappa_n})^2 \Big),$$

where c is a positive constant.

3 - **Error**

The combination of the concentration error and the velocity error gives us the result using the Gronwall Lemma.