Multi-scale finite element methods for advection-diffusion problems

SMAI 2021

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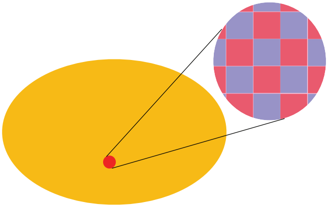
21 June, 2021

A multi-scale diffusion problem

$\Omega \subset \mathbb{R}^d$ a bounded polygon.
$A^\epsilon : \Omega \to \mathbb{R}^{d\times d}$ bounded and uniformly elliptic.
Source term $f \in L^2(\Omega)$.



Search $u^\epsilon \in H^1_0(\Omega)$ solution to the BVP

$$(1) \qquad \begin{cases} -\mathrm{div}(A^\epsilon \nabla u^\epsilon) = f, & \text{in } \Omega, \\ u^\epsilon = 0, & \text{on } \partial\Omega, \end{cases}$$

The parameter ϵ represents one or multiple small scales ($\ll \mathrm{diam}(\Omega)$) in the model.

Conduction/flow in heterogeneous media, composite materials, porous media

For example, A^ϵ might be a rescaling of some (periodic) matrix, i.e., $A^\epsilon(x) = A_{(\mathrm{per})}(x/\epsilon)$.

Finite element approximation

The PDE (1) admits an equivalent variational formulation:

$$\text{Find } u^\epsilon \in H_0^1(\Omega) \text{ such that}$$
$$\forall\, v \in H_0^1(\Omega), \quad a^\epsilon(u^\epsilon, v) = F(v),$$

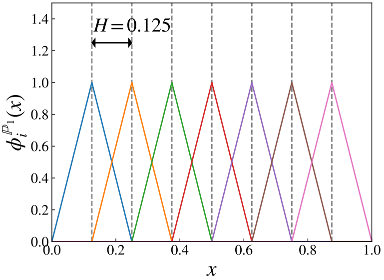
where

$$a^\epsilon(u, v) = \int_\Omega (\nabla v)^\top A^\epsilon \nabla u, \quad F(v) = \int_\Omega fv.$$

Galerkin approximation on $V_H \subset H_0^1(\Omega)$:

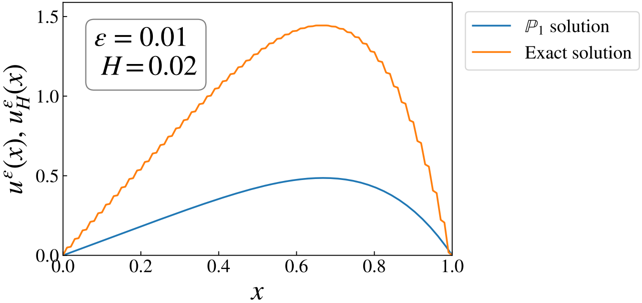
$$\text{Find } u_H^\epsilon \in V_H \text{ such that}$$
$$\forall\, v \in V_H, \quad a^\epsilon(u_H^\epsilon, v) = F(v),$$

We use the $\mathbb{P}_1$ FE method on a mesh $\mathcal{T}_H$:
$V_H = \operatorname{span}\{\phi_i^{\mathbb{P}_1}\}$.



The FEM & multi-scale problems

Consider the example $A^\epsilon(x) = \begin{cases} 1 & \text{if } \lfloor x/\epsilon \rfloor \text{ is even} \\ 10 & \text{if } \lfloor x/\epsilon \rfloor \text{ is odd} \end{cases}$ and $f(x) = Cx^3$ for a mesh of size $H = 2\epsilon$.



The $\mathbb{P}_1$ FEM fails to even capture the macroscopic properties of the solution because the micro-structure is not adequately dealt with – the mesh is too coarse.

FEM on a coarse mesh: MsFEM

On a mesh $\mathcal{T}_H$ of size $H < \epsilon$, the FEM becomes prohibitively expensive.

The multi-scale finite element method (Hou and Wu 1997; Efendiev and Hou 2009) consists of

- Offline stage – multi-scale basis functions ϕ_i^ϵ adapted to A^ϵ:

$$\begin{cases} -\mathrm{div}(A^\epsilon \nabla \phi_i^\epsilon) = 0, & \text{in } K, \\ \phi_i^\epsilon = \phi_i^{\mathbb{P}_1}, & \text{on } \partial K \end{cases} \quad \text{for each mesh element } K \in \mathcal{T}_H. \tag{2}$$

- Online stage – solve the Galerkin approximation

$$\text{Find } u_H^\epsilon \in V_H^\epsilon \quad \text{such that} \quad \forall\, v \in V_H^\epsilon, \quad a^\epsilon(u_H^\epsilon, v) = F(v),$$

where

$$V_H^\epsilon = \mathrm{span}\{\phi_i^\epsilon\}. \tag{3}$$

The number of degrees of freedom in the online stage is the same as for a $\mathbb{P}_1$ method on a coarse mesh.

Advection-dominated problems

Now consider, for some $b \in \mathbb{R}^d$ the BVP

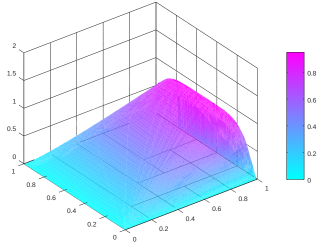
$$(4) \qquad \begin{cases} -\mathrm{div}(A^\epsilon \nabla u^\epsilon) + b \cdot \nabla u^\epsilon = f, & \text{in } \Omega, \\ u^\epsilon = 0, & \text{on } \partial\Omega, \end{cases}$$

When $|b|$ is large with respect to A^ϵ, boundary layers appear in u^ϵ. If

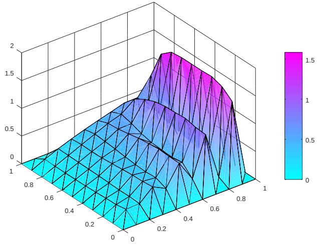
$$\mathrm{Pe} \leftarrow \frac{|b|H}{2 \min A^\epsilon} > 1,$$

even for $A^\epsilon = \text{const.} = 5\mathrm{e}^{-3}$, the $\mathbb{P}_1$ FEM fails ($b = (1,0)^\top$, $f = 1$):

$\mathbb{P}_1$ solution, $\mathrm{Pe} = 0.5\,(H = 0.01)$



$\mathbb{P}_1$ solution, $\mathrm{Pe} = 5\,(H = 0.1)$



Variational formulation

The PDE (4) again admits an equivalent variational formulation:

$$\text{Find } u^\epsilon \in H_0^1(\Omega) \text{ such that}$$
$$\forall\, v \in H_0^1(\Omega), \quad a^\epsilon(u^\epsilon, v) = F(v),$$

where, from now on, we use

$$a^\epsilon(u, v) = \int_\Omega (\nabla v)^\top A^\epsilon \nabla u + vb \cdot \nabla u, \quad F(v) = \int_\Omega fv.$$

Stabilization

Several stabilization approaches for non-multi-scale problems have been proposed:

- SUPG: Streamline-Upwind/Petrov-Galerkin method (Brooks and Hughes 1982);
- Other strongly consistent and/or Petrov-Galerkin methods (Mizukami and Hughes 1985; Hughes, Franca, and Hulbert 1985);
- Adding bubble functions (Baiocchi and Brezzi 1993).

For the stabilization of multi-scale problems, we can also consider

- Advection-adapted basis functions (Park and Hou 2004);

We also mention LOD-type stabilization (Li, Peterseim, and Schedensack 2017).

MsFEM-SUPG (Le Bris, Legoll, and Madiot 2017)

The standard SUPG stabilization can be applied to the MsFEM of (2)-(3):

$$\text{Find } u_H^\epsilon \in V_H^\epsilon \text{ such that } \forall\, v \in V_H^\epsilon : \quad a_H^{\epsilon,SUPG}(u_H^\epsilon, v) = F_H(v),$$

where

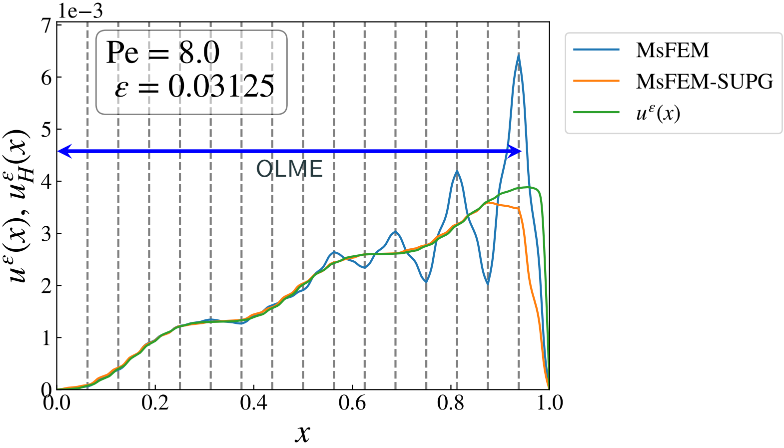
$$a_H^{\epsilon,SUPG}(u,v) = a^\epsilon(u,v) + \sum_{K \in \mathcal{T}_H} \int_K \tau_K\, (b \cdot \nabla u)(b \cdot \nabla v),$$

and

$$F_H(v) = F(v) + \sum_{K \in \mathcal{T}_H} \int_K f\, \tau_K\, b \cdot \nabla v.$$

The choice of the stabilization parameter τ_K is delicate and often inspired by a simple 1-dimensional case (John and Knobloch 2007).

MsFEM-SUPG illustration



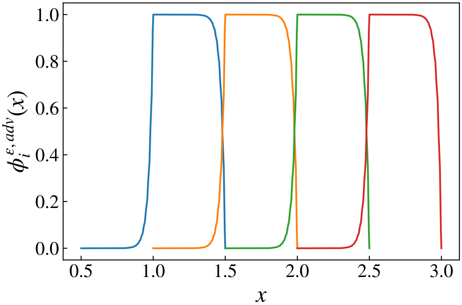
MsFEM-SUPG yields a reasonably accurate solution, but only Outside the Last Mesh Element (of course). ($A^\epsilon(x) = 2 + \cos(2\pi x/\epsilon)$, $\epsilon = 2^{-5}$)

Advection built into the basis functions

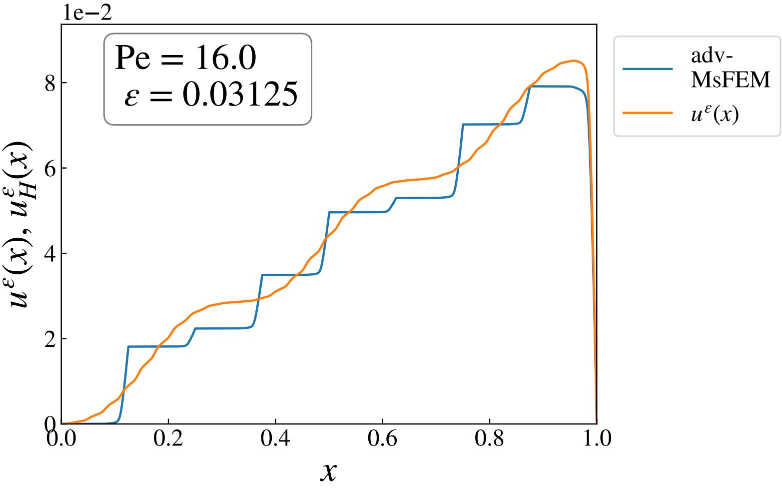
In the spirit of the MsFEM, one can alternatively build basis functions $\phi_i^{\epsilon,\mathrm{adv}}$ that solve the PDE (4) locally, including the advection (Park and Hou 2004; Le Bris, Legoll, and Madiot 2017):

$$\begin{cases} -\mathrm{div}(A^\epsilon \nabla \phi_i^{\epsilon,\mathrm{adv}}) + b \cdot \nabla \phi_i^{\epsilon,\mathrm{adv}} = 0, & \text{in } K, \\ \phi_i^{\epsilon,\mathrm{adv}} = \phi_i^{\mathbb{P}_1}, & \text{on } \partial K, \end{cases} \quad \text{for each } K \in \mathcal{T}_H.$$

We call this the adv-MsFEM.



adv-MsFEM illustration



The adv-MsFEM is stable, but the influence of the advection b on $\phi_i^{\epsilon,\text{adv}}$ seems too strong when Pe is large. $(A^\epsilon(x) = 2 + \cos(2\pi x/\epsilon),\ \epsilon = 2^{-5})$

Bubbles

The interpolation error e for adv-MsFEM

$$e = u^\epsilon - \sum_i u^\epsilon(x_i)\phi_i^{\epsilon,\mathsf{adv}}$$

satisfies (in 1D)

$$\begin{cases} -\mathrm{div}(A^\epsilon \nabla e) + b \cdot \nabla e = f, \text{ in } K, \\ \hfill e = 0, \text{ on } \partial K, \end{cases} \quad \text{for each } K \in \mathcal{T}_H.$$

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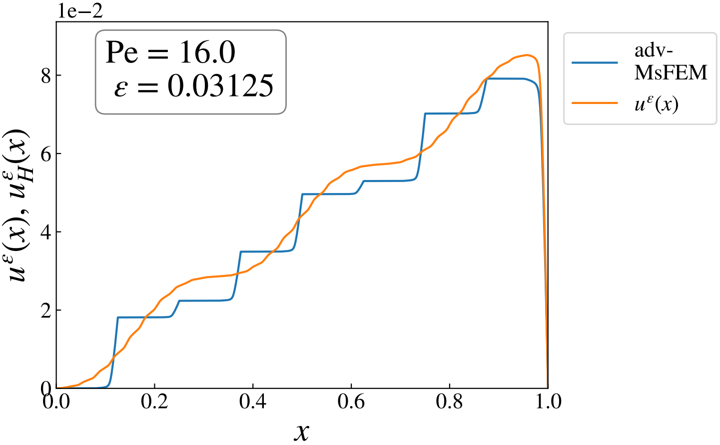
To decrease this error, introduce bubble functions $\phi_K^{\epsilon,\mathrm{adv},B}$ (Biezemans, PhD thesis, in preparation): for each $K \in \mathcal{T}_H$,

$$\begin{cases} -\mathrm{div}(A^\epsilon \nabla \phi_K^{\epsilon,\mathrm{adv},B}) + b \cdot \nabla \phi_K^{\epsilon,\mathrm{adv},B} = 1, \text{ in } K, \\ \qquad \phi_K^{\epsilon,\mathrm{adv},B} = 0, \text{ on } \partial K. \end{cases}$$

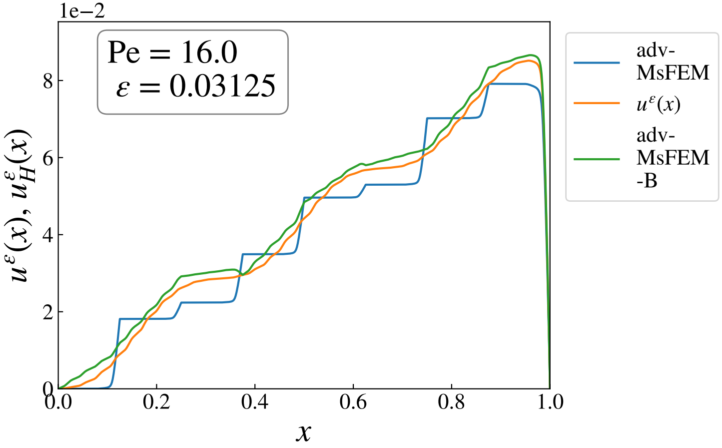
We set $V_H^{\epsilon,\mathrm{adv},B} = \mathrm{span}\{\phi_i^{\epsilon,\mathrm{adv}}, \phi_K^{\epsilon,\mathrm{adv},B}\}$. Then the adv-MsFEM-B reads:

$$\text{Find } u_H^\epsilon \in V_H^{\epsilon,\mathrm{adv},B} \text{ s.t } \forall\, v \in V_H^{\epsilon,\mathrm{adv},B} : a^\epsilon(u_H^\epsilon, v) = F(v).$$

adv-MsFEM and adv-MsFEM-B



adv-MsFEM and adv-MsFEM-B



Remark: Petrov-Galerkin variants of these methods can be used (exactness at the nodes of the mesh in 1D).

On the bubbles in adv-MsFEM-B

Let u_H^ϵ be the adv-MsFEM-B solution. The coefficients β_K in

$$u_H^\epsilon = \sum_{i=1}^{N_{\text{nodes}}} \alpha_i \phi_i^{\epsilon,\text{adv}} + \sum_{K \in \mathcal{T}_H} \beta_K \phi_K^{\epsilon,\text{adv},B},$$

are explicit:

$$\beta_K = \frac{\int_K f \phi_K^{\epsilon,\text{adv},B}}{\int_K \phi_K^{\epsilon,\text{adv},B}} \longrightarrow \beta_K \approx \frac{1}{|K|} \int_K f.$$

A less intrusive variant

$\rightarrow$ A new variant: we look for u_H^ϵ as

$$\sum_{i=1}^{N_{\text{nodes}}} \alpha_i \phi_i^{\epsilon,\text{adv}} + \sum_{K \in \mathcal{T}_H} \left(\frac{1}{|K|} \int_K f \right) \phi_K^{\epsilon,\text{adv},B}, \quad \alpha_i \in \mathbb{R}.$$

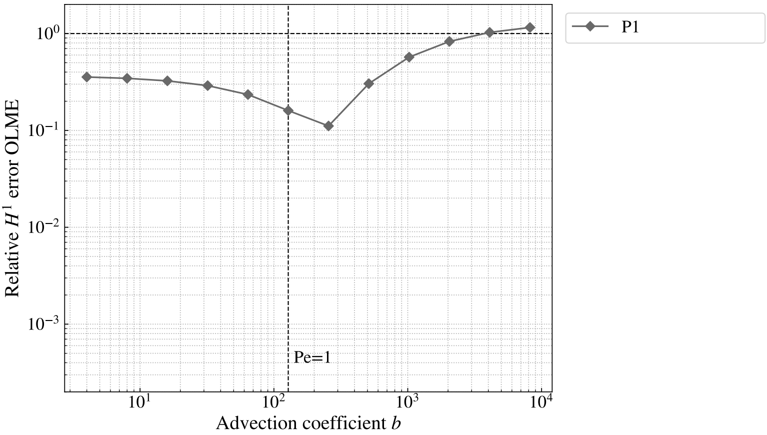
We test against the $\phi_i^{\mathbb{P}_1}$ basis functions to find the α_i.

- This yields a less intrusive method. We call it MsFEM-nonI-B here.
- When f is constant on each K, this is equivalent to a method with adjoint residual-free bubbles (Franca and Russo 2000). This method is exact in 1D.
- Bubble functions come at minimal additional cost in the online computations;

Comparison (1D tests)

We only measure errors outside the last mesh element (OLME).

$$(\text{Relative } H^1 \text{ error})^2 = \frac{\|u^\epsilon - u^\epsilon_H\|^2_{L^2(\text{OLME})} + \|(u^\epsilon - u^\epsilon_H)'\|^2_{L^2(\text{OLME})}}{\|u^\epsilon\|^2_{L^2([0,1])} + \|(u^\epsilon)'\|^2_{L^2([0,1])}}.$$



Test case: $A^\epsilon(x) = 2 + \cos(2\pi x/\epsilon)$, $\epsilon = 2^{-8}$, $H = 2^{-6}$, $f(x) = \sin(3\pi x)^2$.

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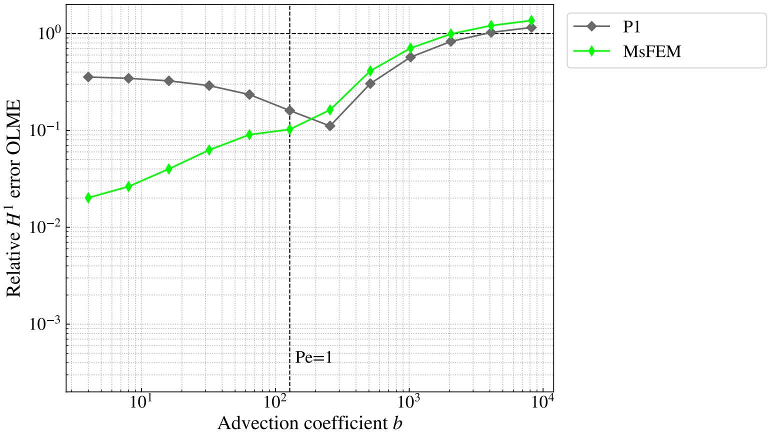

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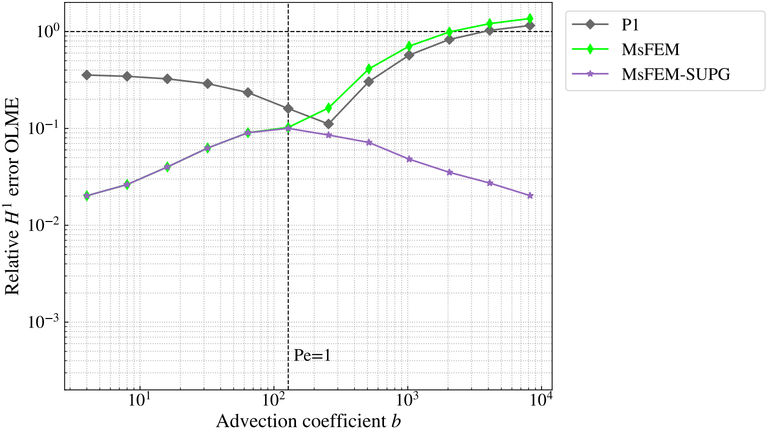

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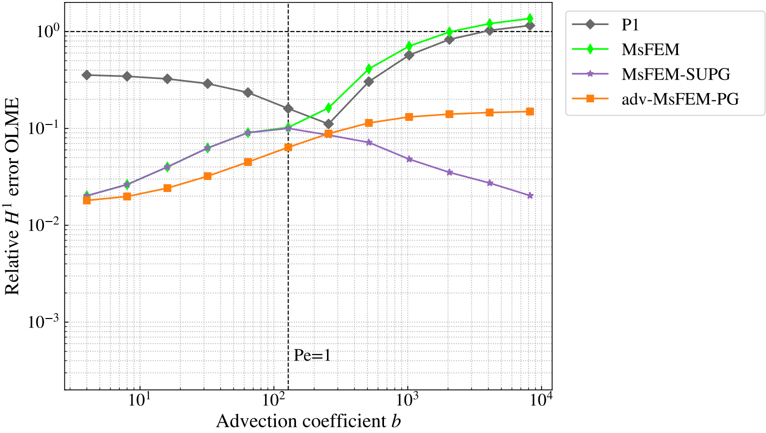

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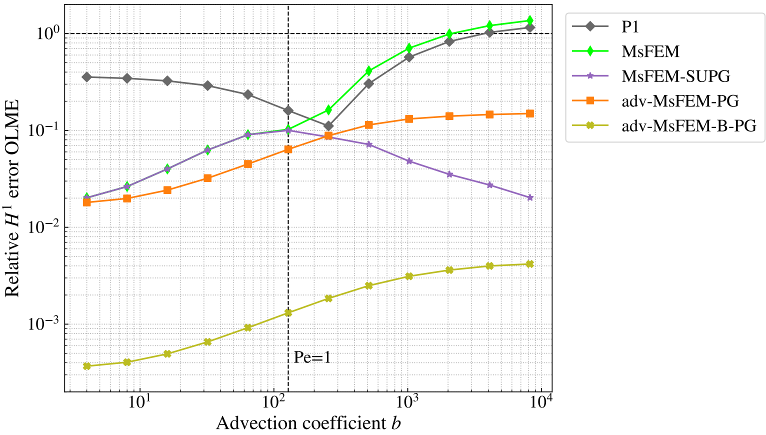

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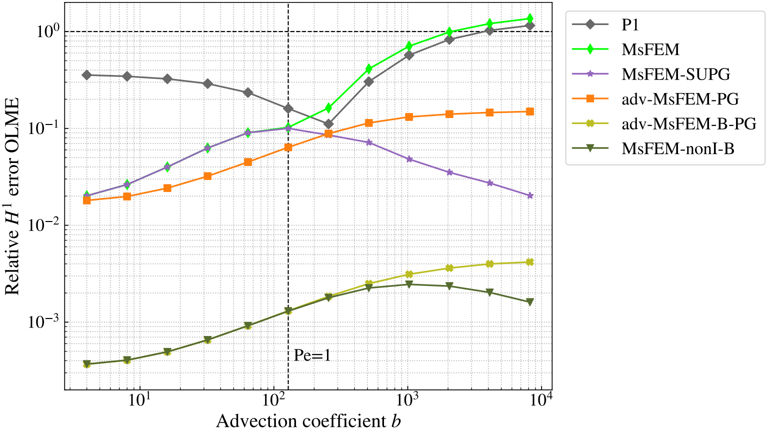

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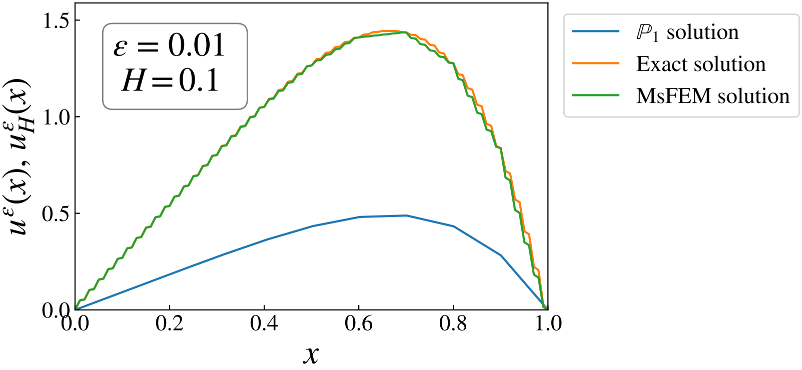
Conclusion and outlook

- The MsFEM can be stabilized by standard non-multi-scale techniques to deal with the advection-dominated regime. This yields good results outside the last mesh element.
- Stabilization can also be achieved by adapting the basis functions (adv-MsFEM). Bubble functions can be added for improved accuracy.
- The best results are obtained with a novel framework for bubble functions, which is also less intrusive with minimal extra online cost.

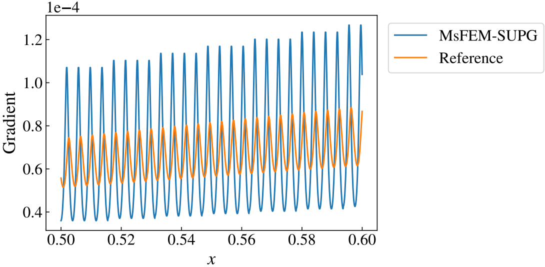
Coming up soon: comparison of the methods for 2D problems.

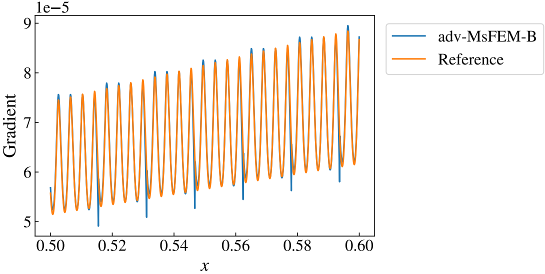
FEM vs MsFEM

Example of slide 3, $H = 10\epsilon$.



Comparison of the derivatives





MsFEM-SUPG:
the effect of b on the amplitude of the oscillations of u^ϵ is not incorporated in the ϕ_i^ϵ.
$(\epsilon = 2^{-8},\ b = 2^{13})$

Additional material - adv-MsFEM and Petrov-Galerkin

For the adv-MsFEM, let us also consider test functions $\psi_i^{\epsilon,\text{adv}}$ solving the adjoint problem locally:

$$(5)\quad \begin{cases} -\text{div}(A^\epsilon \nabla \psi_i^{\epsilon,\text{adv}}) - b \cdot \nabla \psi_i^{\epsilon,\text{adv}} = 0, & \text{in } K, \\ \psi_i^{\epsilon,\text{adv}} = \phi_i^{\mathbb{P}_1}, & \text{on } \partial K \text{ for each } K \in \mathcal{T}_H. \end{cases}$$

Then we can define the following Petrov-Galerkin approximation to (4):

$$(6)\qquad \text{Find } u_H^\epsilon \in V_H^\epsilon \text{ s.t. for each } \psi_i^{\epsilon,\text{adv}} : a^\epsilon(u_H^\epsilon, \psi_i^{\epsilon,\text{adv}}) = F(\psi_i^{\epsilon,\text{adv}}).$$

This adv-MsFEM-PG variant is in dimension 1 exact at the nodes of the mesh, hence must be stable.
The adv-MsFEM has the same stiffness matrix and is thus also stable.

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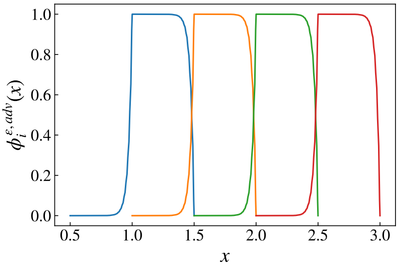

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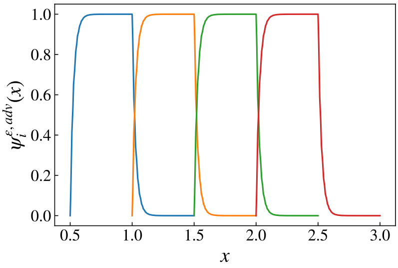

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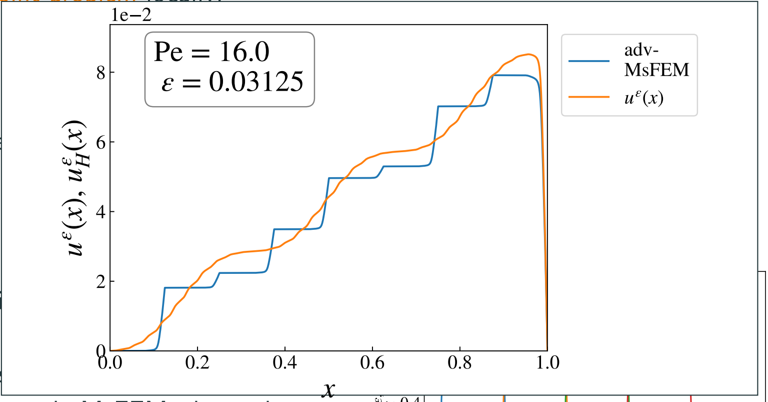

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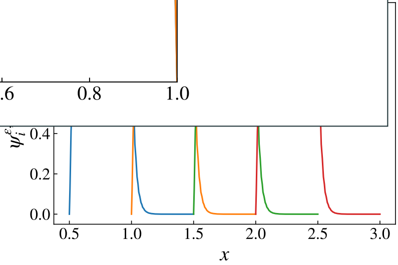

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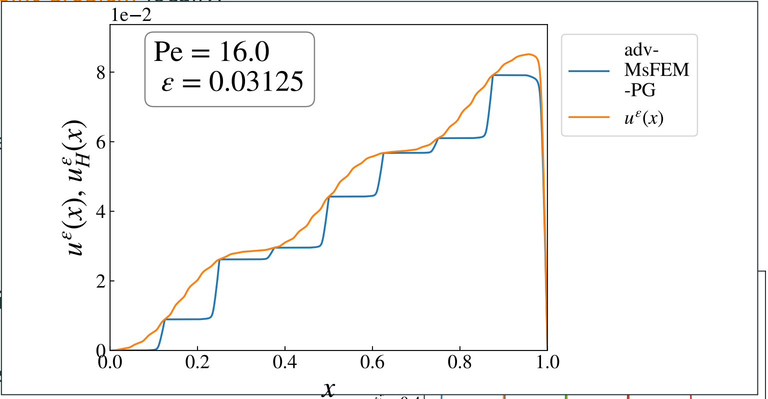

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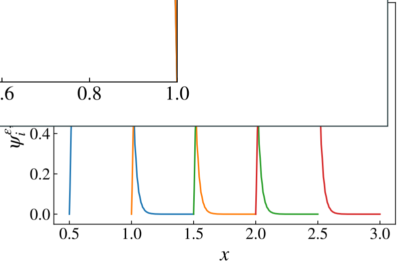

Errors on the entire domain

Test case: $A^\epsilon(x) = 2 + \cos(2\pi x/\epsilon)$, $\epsilon = 2^{-8}$, $H = 2^{-6}$, $f(x) = \sin(3\pi x)^2$.