Stability of finite difference schemes approximations for hyperbolic boundary value problem in an interval.

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The continuous problem

We want to give numerical approximate solutions of **the strip** problem

$$\begin{cases} L(\partial)u := \partial_t u + \sum_{j=1}^{d} A_j \partial_j u = f & \text{for } (t,x) \in [0,T] \times \mathbb{R}^{d-1} \times]0,1[:= \Omega_T, \\ B_0 u_{|x_d=0} = g_0 & \text{on } [0,T] \times \mathbb{R}^{d-1} := \partial_0 \Omega_T, \\ B_1 u_{|x_d=1} = g_1 & \text{on } [0,T] \times \mathbb{R}^{d-1} := \partial_1 \Omega_T, \\ u_{|t=0} = u_0 & \text{on } \mathbb{R}^{d-1} \times]0,1[:= \Gamma. \end{cases} \quad (1)$$

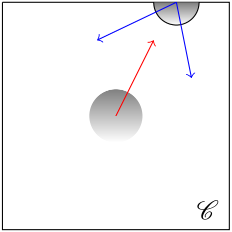
Where $L(\partial)$ is **constantly hyperbolic** with respect to the direction t. The $A_j \in \mathbf{M}_{N\times N}(\mathbb{R})$ so that (1) is a **system** of PDE. $\det A_d \neq 0$, $A_j = A_j^T$.

Classical framework of **wave propagation phenomena**, *waves, Maxwell, linearisation of Euler...*

The **generic** boundary matrices $B_0 \in \mathbf{M}_{p\times N}(\mathbb{R})$, $B_1 \in \mathbf{M}_{(N-p)\times N}(\mathbb{R})$ encode **the good number of boundary conditions**.

Motivations

- **Transparent/absorbent boundary conditions** for the implementation of the Cauchy problem.



$\mathbb{R}^d$

- The corner problem is too difficult. Not a lot of progress since [Osher '73]. We keep the difficulty of two boundary conditions but simpler geometry.

Well-posedness of the continuous problem

Definition (Well-posedness)

For all sources $f \in L^2_\gamma(\Omega_T)$, $g_0 \in L^2_\gamma(\partial_0\Omega_T)$, $g_1 \in L^2_\gamma(\partial_1\Omega_T)$, $u_0 \equiv 0$ there exists **a unique** solution $u \in L^2_\gamma(\Omega_T)$ of (1) with traces in $L^2_\gamma(\partial\Omega_T)$ satisfying : $\exists\, \gamma_0 \geq 0$ such that $\forall \gamma > \gamma_0$ we have

$$\gamma \|u\|^2_{L^2_\gamma(\Omega_T)} + \|u_{|x_d=0}\|^2_{L^2_\gamma(\partial_0\Omega_T)} + \|u_{|x_d=1}\|^2_{L^2_\gamma(\partial_1\Omega_T)} \lesssim \frac{1}{\gamma}\|f\|^2_{L^2_\gamma(\Omega_T)} + \|g_0\|^2_{L^2_\gamma(\partial_0\Omega_T)} + \|g_1\|^2_{L^2_\gamma(\partial_1\Omega_T)}. \tag{2}$$

Or if $u_0 \not\equiv 0$, $\forall T > 0$

$$e^{-2\gamma T}\|u(T,\cdot)\|^2_{L^2(\Gamma)} + \gamma \|u\|^2_{L^2_\gamma(\Omega_T)} + \|u_{|x_d=0}\|^2_{L^2_\gamma(\partial_0\Omega_T)} + \|u_{|x_d=1}\|^2_{L^2_\gamma(\partial_1\Omega_T)} \lesssim \frac{1}{\gamma}\|f\|^2_{L^2_\gamma(\Omega_T)} + \|g_0\|^2_{L^2_\gamma(\partial_0\Omega_T)} + \|g_1\|^2_{L^2_\gamma(\partial_1\Omega_T)} + \|u_0\|^2_{L^2(\Gamma)}. \tag{3}$$

where

$$\|\cdot\|_{L^2_\gamma} := \|e^{-\gamma t}\cdot\|_{L^2},$$

Well-posedness of the continuous problem, result 1

Remarks :

- **Straightforward** generalization of the concept of well-posedness in the **half-space** [Kreiss '70].
- Because possibly $\gamma_0 > 0$, possible **exponential growth** with t ruled by $e^{\gamma_0 t}$. In the half-space $\gamma_0 = 0$ **sharp well-posedness**=**lower exponential growth in time**.

Aim : **Characterize** the boundary matrices B_0, B_1 such that the problem is **sharply (or not) well-posed**.

Proposition

If B_0 (resp. B_1) is such that the problem in the half-space $\{x_d > 0\}$ (resp. $\{x_d < 1\}$ is **sharply** *well-posed then the strip problem (1) is well-posed with $\gamma_0 > 0$.*

<u>**Proof**</u> : **Localisation** one interior problem + 2 boundary problems. **Triangle inequality**.

Well-posedness of the continuous problem, result 2

Previous result **clearly not sharp**. *Does the growth really appear ?*

Theorem (B. '20)

Under structural assumptions, the strip problem (1) *is sharply well-posed* **"if and only if"** *some matrices reading under the form* $I - \mathbf{T}$ *are* **uniformly invertible**.

- **T** depend explicitly on the A_j, B_0, B_1.
- **T** is a **trace operator** that to a trace $u_{|x_d=0}$ associates the trace obtained after the **free evolution to the right and then reflected back to the left**.
- Exponential growth ruled by $e^{\gamma_0 t}$ can **effectively appear** because of **trapped rays** [B. '20].
- **For the continuous problem the solutions with exponential growth in time are characterized**.
- However condition **difficult to check** effectively.

Discrete approximation

Approximation by **finite difference schemes** of

$$\begin{cases} \partial_t u + A\partial_x u = f & \text{for } (t,x) \in [0,T] \times]0,1[, \\ B_0 u_{|x=0} = g_0 & \text{on } [0,T], \\ B_1 u_{|x=1} = g_1 & \text{on } [0,T], \\ u_{|t=0} = u_0 & \text{on }]0,1[. \end{cases}$$

- For the half-line characterization of stable schemes [GKS '72].
- $d > 1$ **widely open question** even for $x_d > 0$ [Michelson '83]
- **Interval** : little results in the literature [GKS '72] and [Trefethen '85] but for **particular** schemes and **particular** boundary conditions.
- **Aim : characterize the stability** for the **most generic class of scheme/boundary conditions possible**.

The associated scheme I : the interior

Consider (x_j) $j \in [\![0, K]\!]$ a **regular subdivision** of $[0, 1]$ and the approximation scheme in the interior

$$\begin{cases} U_j^{n+1} + QU_j^n = \Delta t F_j^{n+1} & \text{for } n \geq 0,\ j \in [\![1, K]\!], \\ U_j^0 = u_j^0, & \text{for } j \in [\![1-\ell, K+r]\!], \end{cases} \tag{4}$$

where for $(\mathbf{T}_j U)_j := U_{j+1}$ the right shift operator

$$Q := \sum_{\mu=-\ell}^{r} A^\mu \mathbf{T}_j^\mu, \;\; A^\mu \in \mathbf{M}_{N\times N}(\mathbb{R}).$$

One step in time , $(\ell + r)$ in **space** (ℓ on the "left" , r on the "right").

- Lax-Friedrichs $A^{-1} = -\frac{1}{2}(I + \lambda A)$, $A^0 = 0$, $A^1 = -\frac{1}{2}(I - \lambda A)$
- Lax-Wendroff $A^{-1} = \frac{\lambda A}{2}(I - \lambda A)$, $A^0 = -I + \lambda^2 A^2$, $A^1 = -\frac{\lambda A}{2}(I + \lambda A)$

The associated scheme II : the boundary conditions

Need **artificial** boundary conditions to determine the U_j , $j \in [\![1-\ell, 0]\!] \cup [\![K+1, K+r]\!]$.

Let

$$\begin{cases} U_j^{n+1} + B_{0,j} U_1^n = G_{0,j}^{n+1} & n \geq 0,\ j \in [\![1-\ell, 0]\!], \\ U_j^{n+1} + B_{1,j}^j U_K^n = G_{1,j}^{n+1} & n \geq 0,\ j \in [\![K+1, K+r]\!], \end{cases} \tag{5}$$

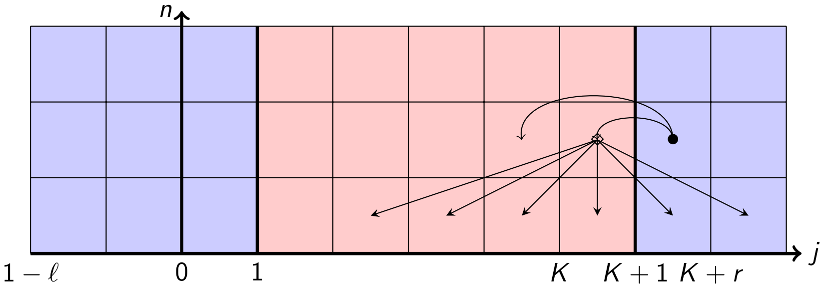
where with $(\mathbf{T}_n U)_n := U^{n+1}$:

$$B_{0,j} := \sum_{\sigma=-1}^{0} \sum_{\mu=0}^{b_0} B_{0,j}^{\sigma,\mu} \mathbf{T}_n^{\sigma} \mathbf{T}_j^{\mu} \text{ and } B_{1,j} := \sum_{\sigma=-1}^{0} \sum_{\mu=-b_1}^{0} B_{1,j}^{\sigma,\mu} \mathbf{T}_n^{\sigma} \mathbf{T}_j^{\mu}.$$

b_0 steps for the left boundary, b_1 for the right.

Illustration

$\ell = 3,\ r = 2$ and $b_1 = 1.$



A main difference between the continuous and the discrete setting is that the continuous boundary conditions prescribe p (or $N - p$) components of the trace(s). Whereas the discrete scheme prescribes the N components.

Need **extra boundary conditions** which are not given by the physics of the continuous problem. **Arbitrary choices**.

Definition of the stability

Definition (Strong stability)

Let $\gamma > \gamma_0$ the scheme (4)-(5) is said to be strongly stable if for all source terms

- $u_j^0 \equiv 0$ **strong/sharp** ($\gamma_0 = 0$) stability

$$\frac{\gamma}{\gamma+1}\sum_{n\geq 1}\sum_{j=1-\ell}^{K+r}|e^{-\gamma n}U_j^n|^2 + \sum_{n\geq 1}\sum_{j=1-\ell}^{r}|e^{-\gamma n}U_j^n|^2 + \sum_{n\geq 1}\sum_{j=K-\ell}^{K+r}|e^{-\gamma n}U_j^n|^2$$

$$\lesssim \frac{\gamma+1}{\gamma}\sum_{n\geq 1}\sum_{j=1}^{K}|e^{-\gamma n}F_j^n|^2 + \sum_{n\geq 1}\sum_{j=1-\ell}^{0}|e^{-\gamma n}G_{0,j}^n|^2 + \sum_{n\geq 1}\sum_{j=K+1}^{K+r}|e^{-\gamma n}G_{1,j}^n|^2.$$

- $u_j^0 \not\equiv 0$ **semigroup** stability. Same estimate with $\sup_n e^{-2\gamma n}\sum_{j=1-\ell}^{K+r}|U_j^n|^2$ in the LHS and $\sum_{j=1-\ell}^{K+r}|u_j^0|^2$ in the RHS

Just **discrete versions of the energy estimates** in the continuous setting.

Discrete problem in a nutshell

We consider the finite difference scheme approximation in the half-line $[0,\infty[$

$$\begin{cases} U_j^{n+1} + QU_j^n = \Delta t F_j^{n+1} & n \geq 0,\ j \geq 1, \\ U_j^{n+1} + B_j U_1^n = G_j^{n+1} & n \geq 0,\ j \in [\![1-\ell, 0]\!], \\ U_j^0 = 0, & j \in [\![1-\ell, \infty[\![\end{cases} \tag{6}$$

We perform a discrete Laplace transform by setting for $z \in \mathbb{C} \setminus \{0\}$, $U_j^n = z^n V_j$. So (6) becomes **(at the formal level)**:

$$\begin{cases} V_j + \frac{1}{z} QV_j = \widetilde{F}_j & j \in [\![1, \infty]\!], \\ V + \frac{1}{z} B_j V_1 = \widetilde{G}_j & j \in [\![1-\ell, 0]\!], \end{cases}$$

and can be rewritten in terms of the augmented vector $\mathscr{V}_j := (V_{j+r-1}, ..., V_{j-\ell})$ under **the purely resolvent** form

$$\begin{cases} \mathscr{V}_{j+1} = \mathbb{M}(z)\mathscr{V}_j + \mathscr{F}_j & j \in [\![1, \infty]\!], \\ \mathbb{B}(z)\mathscr{V}_1 = \mathscr{G}_j & j \in [\![1-\ell, 0]\!], \end{cases} \tag{7}$$

where

$$\mathbb{M}(z) := \begin{bmatrix} -(\mathbb{A}_r)(z)^{-1}\mathbb{A}_{r-1}(z) & \cdots & \cdots & (\mathbb{A}_r)(z)^{-1}\mathbb{A}_{\ell}(z) \\ I & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix} \in \mathbf{M}_{(\ell+r)N\times(\ell+r)N}(\mathbb{C}),$$

with

$$\forall z \in \mathbb{C}\setminus\{0\}\,,\ \forall \mu \in [\![-\ell, r]\!],\ \mathbb{A}_{\mu}(z) := \delta_{\mu,0} I - \frac{1}{z}\ A^{\mu}.$$

Proposition

For all $z \in \mathbb{C}$, such that $|z| > 1$, the eigenvalues $\lambda(z)$ of $\mathbb{M}(z)$ satisfy

- $|\lambda(z)| < 1$. *We denote by $\mathbb{E}^s(z)$ the associated generalized eigenspace.*
- $|\lambda(z)| > 1$. *We denote by $\mathbb{E}^u(z)$ the associated generalized eigenspace.*

Sharp stability result in the half-line

Theorem ("[GKS '73]")

The scheme in the **half-line** **is sharply stable if and only the so-called GKS condition** *holds that is*

$$\forall z \in \mathbb{C},\ \text{s.t. } |z| \geq 1,\ \text{we have } \mathbb{E}^s(z) \cap \ker \mathbb{B} = \{0\}\,.$$

If the **segment** *problem satisfies the GKS condition on each side that is*

$$\forall z \in \mathbb{C},\ \text{s.t. } |z| \geq 1,\ \text{we have } \mathbb{E}^s(z) \cap \ker \mathbb{B}_0(z) = \{0\} = \mathbb{E}^u(z) \cap \ker \mathbb{B}_1(z),$$

then the segment problem is **stable** *with* $\gamma_0 > 0$.

The question of sharp stability

Problem : Compared to the half-line scheme the segment scheme may admit a solution **with non-trivial growth compared to time**.

"Does **a bad choice of the extra boundary conditions** can give a non trivially growing approximation while the solution of the continuous problem does not and *vice versa*?"

Need to characterize the non-trivially growing **approximations schemes**

Characterization of sharp stability

Theorem (B. Preprint)

Under structural assumptions, there exists a matrix $\mathbb{T}(z)$ such that :

- *If the finite difference approximation scheme is sharply stable then $I - \mathbb{T}(z)$ is* **invertible***.*
- *If $I - \mathbb{T}(z)$ is* **uniformly invertible** *then the finite difference approximation scheme is sharply stable.*

We have for free the same result as in the half-line [Coulombel-Gloria '10]

Theorem (B. Preprint)

Under **the same structural assumption** *if the finite difference approximation scheme is* **sharply stable** *then it is also* **semigroup stable***.*

Some remarks on the result

- The matrix $\mathbb{T}$ is just a **discretized version** of $\mathbf{T}$. Same **continuous to discrete extension condition** than UKL condition for the half-line.
- We recover the best possible stability result (semigroup) with no more restriction than in the half-line geometry [Coulombel-Gloria '10].
- The proof relies on the adaptation to the discrete setting of some ideas introduced to deal with the **corner problem** [Osher '73].

Thank you for your attention.