

A fully well-balanced scheme for shallow water equations with Coriolis force

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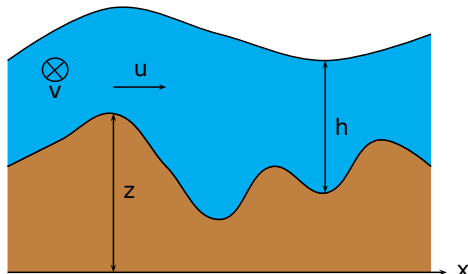


Introduction

Shallow water equations with Coriolis force :

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + \frac{gh^2}{2}\right) = -gh\partial_x z + fhv, \\ \partial_t(hv) + \partial_x(huv) = -fhu. \end{cases}$$

- $h(x, t)$: fluid height
- $u(x, t)$: tangential velocity
- $v(x, t)$: transverse velocity
- $z(x)$: topography
- f : Coriolis force
- g : gravity constant



Notations : $w = (h, hu, hv)^T$, $\tilde{w} = (w, z)$.

Existing works

With Coriolis force : steady states at rest preserved

[Bouchut et al., 2004],
[Lukáčová-Medvid'ová et al., 2007],
[Audusse et al., 2011],
[Chertock et al., 2018].

Goals :

- preserve positivity of h ,
- get a fully well-balanced scheme (FWB), [Noelle et al., 2007], [Berthon and Chalons, 2016], [Berthon et al., 2021],
- increase precision, [Michel-Dansac et al., 2016], [Ghitti et al., 2020].

Outline

- 1 A fully well-balanced Godunov-type scheme
- 2 Second-order extension
- 3 Numerical results

Outline

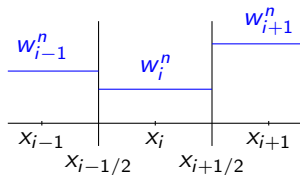
1 A fully well-balanced Godunov-type scheme

2 Second-order extension

3 Numerical results

Godunov-type scheme principle

Finite volume discretisation at time t^n is a juxtaposition of Riemann problems

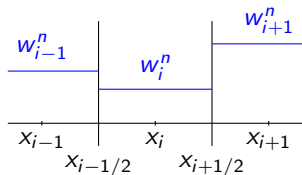


Exact Riemann solver : $\mathcal{W}_R \left(\frac{x}{t}, \tilde{w}_L, \tilde{w}_R \right)$

Approximate Riemann solver : $\widehat{\mathcal{W}}_R \left(\frac{x}{t}, \tilde{w}_L, \tilde{w}_R \right)$

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Approximation at time t^{n+1} obtained in two steps :

- $w_{\Delta x}(x, t^n + t) = \widehat{\mathcal{W}}_R \left(\frac{x - x_{i+1/2}}{t}, \tilde{w}_i^n, \tilde{w}_{i+1}^n \right)$ on $[x_i, x_{i+1}]$.
- $w_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} w_{\Delta x}(x, t^n + \Delta t) dx.$

Steady states discretisation

$$\begin{cases} \partial_x(hu) = 0, \\ \partial_x\left(\frac{u^2}{2} + g(h+z)\right) = fv, \\ u\partial_x v = -fu. \end{cases}$$

$$\begin{cases} h_R u_R = h_L u_L = q, \\ \left[\frac{u^2}{2} + g(h+z)\right] = \Delta x f \bar{v}, \\ q[v] = -\Delta x f q, \end{cases}$$



discretisation

$$[X] = X_R - X_L, \quad \bar{X} = \frac{X_L + X_R}{2}.$$

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discretisation

Steady state indicator :

$$\mathcal{E}(\tilde{w}_L, \tilde{w}_R, \Delta x) := \sqrt{\left| [hu] \right|^2 + \left| \left[\frac{u^2}{2} + g(h+z) \right] - \Delta x f \bar{v} \right|^2 + \left| \overline{hu}([v] + f \Delta x) \right|^2}.$$

Definition

A couple $(\tilde{w}_L, \tilde{w}_R)$ defines a local steady state for the system if $\mathcal{E}(\tilde{w}_L, \tilde{w}_R, \Delta x) = 0$.

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Steady state indicator :

$$\mathcal{E}_{LR} = \sqrt{\left|[hu]\right|^2 + \left|\left[\frac{u^2}{2} + g(h+z)\right] - \Delta x f \bar{v}\right|^2 + \left|\bar{h}u([v] + f\Delta x)\right|^2}.$$

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Approximate Riemann solver

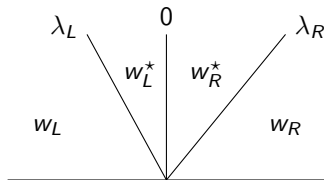


Figure: Approximate Riemann solver $\widehat{\mathcal{W}}_R$

To determine w_L^* and w_R^* (6 unknowns) we use

- the consistency condition (3 relations) :

$$\frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \widehat{\mathcal{W}}_R \left(\frac{x}{\Delta t}, \tilde{w}_L, \tilde{w}_R \right) dx = \frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \mathcal{W}_R \left(\frac{x}{\Delta t}, \tilde{w}_L, \tilde{w}_R \right) dx$$

- the fully well-balanced condition (3 relations):
if $(\tilde{w}_L, \tilde{w}_R)$ is a local steady state then $w_L^* = w_L$ and $w_R^* = w_R$.

Approximate Riemann solver

We set $\alpha_{LR} = g\bar{h} - |u_L u_R|$.

Let $(\tilde{w}_L, \tilde{w}_R)$ be a local steady state. The FWB property implies

$$q = h_L u_L = h_R u_R,$$

$$\alpha_{LR}(h_R - h_L) = S^{hu}(\tilde{w}_L, \tilde{w}_R),$$

$$q(v_R - v_L) = S^{hv}(\tilde{w}_L, \tilde{w}_R).$$

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For any $(\tilde{w}_L, \tilde{w}_R)$, we set

If $\mathcal{E}_{LR} \neq 0$,

$$q^* = h_L^* u_L^* = h_R^* u_R^*,$$

$$h_R^* - h_L^* = \frac{\alpha_{LR} S^{hu}(\tilde{w}_L, \tilde{w}_R)}{\alpha_{LR}^2 + \mathcal{E}_{LR}}$$

$$v_R^* - v_L^* = \frac{\tilde{q} S^{hv}(\tilde{w}_L, \tilde{w}_R)}{\tilde{q}^2 + \mathcal{E}_{LR}}$$

If $\mathcal{E}_{LR} = 0$,

$$q^* = h_L^* u_L^* = h_R^* u_R^*,$$

$$h_R^* - h_L^* = h_R - h_L,$$

$$v_R^* - v_L^* = v_R - v_L.$$

Source term discretisation

Let $(\tilde{w}_L, \tilde{w}_R)$ be a local steady state.

The FWB condition AND the chosen steady state discretisation implies

$$S^{hu}(\tilde{w}_L, \tilde{w}_R) = \Delta x f \bar{h} \bar{v} - g \bar{h} [z] + \frac{g \text{Fr}[h]}{4 \bar{h}} \frac{(\Delta x f \bar{v} / g - [z])^2}{(1 - \text{Fr})^2},$$

$$S^{hv}(\tilde{w}_L, \tilde{w}_R) = -\Delta x f q.$$

where $\text{Fr} = \frac{\bar{h} |u_L u_R|}{g h_L h_R}$ is a discrete Froude number.

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Remark : $\lim_{\text{Fr} \rightarrow 1} S^{hu}(\tilde{w}_L, \tilde{w}_R) = \Delta x f \bar{h} \bar{v} - g \bar{h} [z] + \frac{g}{4\bar{h}} [h]^3.$

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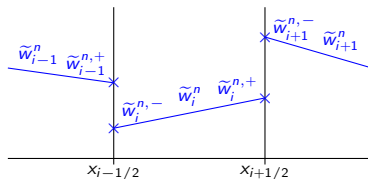
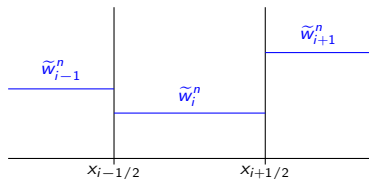
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Piecewise reconstruction



$$\sigma_i^n(\tilde{w}) = \min\left(\frac{\tilde{w}_i^n - \tilde{w}_{i-1}^n}{\Delta x}, \frac{\tilde{w}_{i+1}^n - \tilde{w}_i^n}{\Delta x}\right)$$

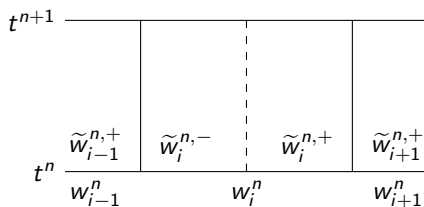
$$\tilde{w}_i^{n,\pm} = \tilde{w}_i^n \pm \frac{\Delta x}{2} \sigma_i^n(\tilde{w})$$

MUSCL method

First-order scheme :

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} (F(\tilde{w}_i^n, \tilde{w}_{i+1}^n, \Delta x) - F(\tilde{w}_{i-1}^n, \tilde{w}_i^n, \Delta x)) \\ + \frac{\Delta t}{2\Delta x} (S(\tilde{w}_{i-1}^n, \tilde{w}_i^n, \Delta x) + S(\tilde{w}_i^n, \tilde{w}_{i+1}^n, \Delta x)).$$

Second-order scheme :

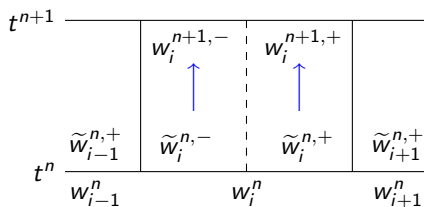


MUSCL method

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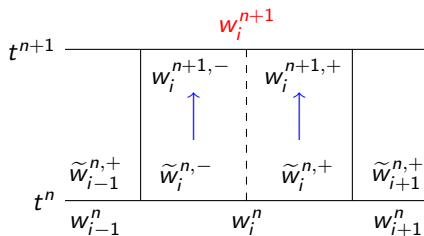


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Second-order scheme :



$$w_i^{n+1} = \frac{1}{2} (w_i^{n+1,-} + w_i^{n+1,+})$$

MUSCL method

Second-order scheme :

$$w_i^{n+1} = \frac{w_i^{n,+} + w_i^{n,-}}{2} - \frac{\Delta t}{\Delta x} \left(F \left(\tilde{w}_i^{n,+}, \tilde{w}_{i+1}^{n,-}, \frac{\Delta x}{2} \right) - F \left(\tilde{w}_{i-1}^{n,+}, \tilde{w}_i^{n,-}, \frac{\Delta x}{2} \right) \right) \\ + \frac{\Delta t}{2\Delta x} \left(S \left(\tilde{w}_{i-1}^{n,+}, \tilde{w}_i^{n,-}, \frac{\Delta x}{2} \right) + 2S \left(\tilde{w}_i^{n,-}, \tilde{w}_i^{n,+}, \frac{\Delta x}{2} \right) + S \left(\tilde{w}_i^{n,+}, \tilde{w}_{i+1}^{n,-}, \frac{\Delta x}{2} \right) \right).$$

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Required reconstruction properties :

- conservative : $\frac{w_i^{n,+} + w_i^{n,-}}{2} = w_i^n$,

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- conservative : $\frac{w_i^{n,+} + w_i^{n,-}}{2} = w_i^n$,
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$$\mathcal{E}(\tilde{w}_i^{n,+}, \tilde{w}_{i+1}^{n,-}, \Delta x/2) = \mathcal{E}(\tilde{w}_i^{n,-}, \tilde{w}_i^{n,+}, \Delta x/2) = 0, \text{ for all } i \in \mathbb{Z}.$$

MUSCL method

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Incompatible conditions

Fully well-balanced recovering

Key idea :

- Consider the second-order scheme far from steady state,
- Use the first-order scheme at steady state.

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New reconstruction :

$$\tilde{w}_i^{n,\pm} = \tilde{w}_i^n \pm \theta_i^n \frac{\Delta x}{2} \sigma_i^n(\tilde{w}),$$

$$\theta_i^n \approx \begin{cases} 0 & \text{at steady state,} \\ 1 & \text{far from a steady state.} \end{cases}$$

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$$\text{with } \Delta x_1 = \Delta x \left(1 - \frac{\theta_i^n}{2}\right) \quad \text{and} \quad \Delta x_2 = \theta_i^n \frac{\Delta x}{2}.$$

Outline

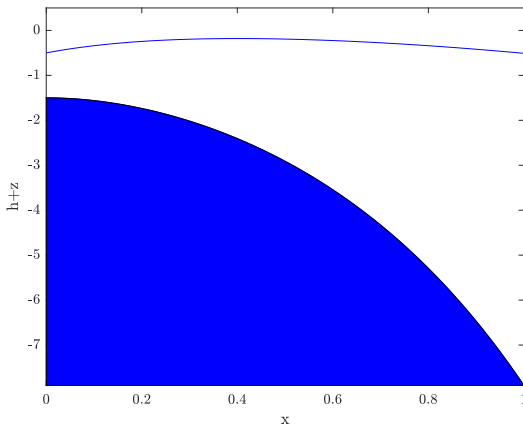
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Numerical results : moving steady state

$$h_0(x) = \exp^{2x}, u_0(x) = \exp^{-2x}, v_0(x) = -fx,$$

$$z(x) = -\frac{1}{2}f^2x^2 - \exp^{2x} - \frac{1}{2}\exp^{-4x},$$

$$f = g = 1.$$



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Steady state distance at time t^n :

$$\mathcal{E}_{\infty,j}^n = \max_{1 \leq i \leq N} \mathcal{E}(\tilde{w}_i^n, \tilde{w}_{i+1}^n, \Delta x).$$

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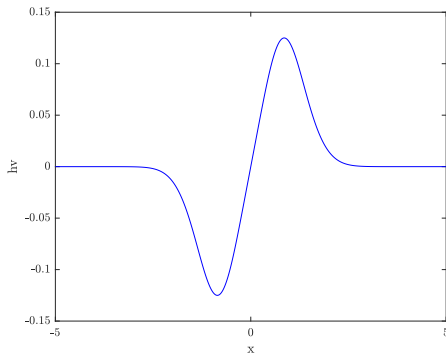
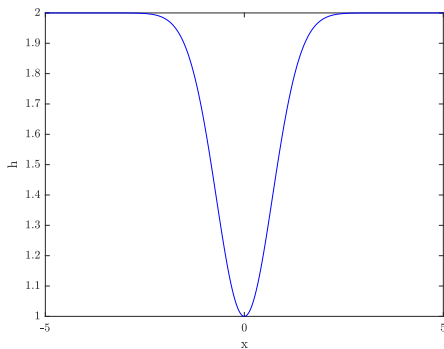
$$T_{\max} = 1, N = 200.$$

- $\mathcal{E}_{\infty,1}^0 = \mathcal{E}_{\infty,2}^0 = 8.87 \times 10^{-16}$
- $\mathcal{E}_{\infty,1}^{T_{\max}} = 5.19 \times 10^{-14}$
- $\mathcal{E}_{\infty,2}^{T_{\max}} = 8.86 \times 10^{-15}$

Numerical results : geostrophic steady state

$$h_0(x) = \frac{2}{g} - e^{-x^2}, u_0(x) = 0, v_0(x) = \frac{2g}{f} x e^{-x^2},$$

$$f = 10, g = 1.$$



Numerical results : geostrophic steady state

$$T_{\max} = 200, N = 200.$$

- $\mathcal{E}_{\infty,1}^0 = \mathcal{E}_{\infty,2}^0 = 4.06 \times 10^{-5}$
- $\mathcal{E}_{\infty,1}^{T_{\max}} = 1.12 \times 10^{-7}$
- $\mathcal{E}_{\infty,2}^{T_{\max}} = 2.53 \times 10^{-12}$

Numerical results : geostrophic steady state

$$T_{\max} = 200, N = 200.$$

- $\mathcal{E}_{\infty,1}^0 = \mathcal{E}_{\infty,2}^0 = 4.06 \times 10^{-5}$
- $\mathcal{E}_{\infty,1}^{T_{\max}} = 1.12 \times 10^{-7}$
- $\mathcal{E}_{\infty,2}^{T_{\max}} = 2.53 \times 10^{-12}$

Table: L^1 -error, first-order scheme

N	h		hv	
200	5.25×10^{-5}		2.11×10^{-4}	
400	1.31×10^{-5}	2.00	5.30×10^{-5}	1.99
800	3.30×10^{-6}	1.99	1.38×10^{-5}	1.94
1600	8.58×10^{-7}	1.94	3.73×10^{-6}	1.88
3200	2.30×10^{-7}	1.91	1.02×10^{-6}	1.87
6400	6.01×10^{-8}	1.93	2.73×10^{-7}	1.90

Table: L^1 -error, second-order scheme

N	h		hv	
200	5.26×10^{-5}		2.11×10^{-4}	
400	1.31×10^{-5}	2.00	5.27×10^{-5}	2.00
800	3.29×10^{-6}	2.00	1.32×10^{-5}	2.00
1600	8.22×10^{-7}	2.00	3.30×10^{-6}	2.00
3200	2.05×10^{-7}	2.00	8.25×10^{-7}	2.00
6400	5.14×10^{-8}	2.00	2.06×10^{-7}	2.00

Perspectives

In sight :

- Considering f variable.
- Build a scheme for the 2D system with Coriolis force.
- Take temperature into consideration.

Perspectives

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Thanks for your attention

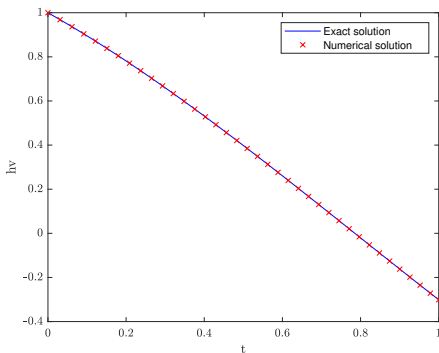
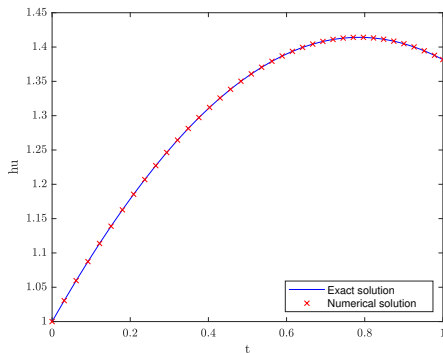
Stationary state in space

For a constant initial condition (h_0, u_0, v_0) fixed, the exact solution of RSW equations writes

$$h(x, t) = h_0,$$

$$u(t) = u_0 \cos(ft) + v_0 \sin(ft),$$

$$v(t) = v_0 \cos(ft) - u_0 \sin(ft).$$



Stationary state in space

$$T_{\max} = 1, N = 200.$$

Table: L^1 -error, first-order scheme

N	hu		hv	
200	3.82×10^{-4}	0.99	8.06×10^{-5}	0.99
400	1.91×10^{-4}	0.99	4.03×10^{-5}	0.99
800	9.56×10^{-5}	0.99	2.01×10^{-5}	0.99
1600	4.78×10^{-5}	0.99	1.01×10^{-5}	0.99
3200	2.39×10^{-5}	0.99	5.04×10^{-6}	0.99
6400	1.20×10^{-5}	0.99	2.52×10^{-6}	0.99

Table: L^1 -error, second-order scheme

N	hu		hv	
200	7.71×10^{-9}	1.99	3.58×10^{-8}	1.99
400	1.92×10^{-9}	1.99	8.95×10^{-9}	1.99
800	4.82×10^{-10}	2.00	2.24×10^{-9}	1.99
1600	1.20×10^{-10}	2.00	5.60×10^{-10}	1.99
3200	3.01×10^{-11}	2.00	1.40×10^{-10}	1.99
6400	7.52×10^{-12}	2.00	3.50×10^{-11}	1.99

Consistency

Consistency condition :

$$\begin{aligned}\frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \widehat{\mathcal{W}}_R \left(\frac{x}{\Delta t}, \tilde{w}_L, \tilde{w}_R \right) dx &= \frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \mathcal{W}_R \left(\frac{x}{\Delta t}, \tilde{w}_L, \tilde{w}_R \right) dx \\ &= \frac{w_L + w_R}{2} - \frac{\Delta t}{\Delta x} (f(w_R) - f(w_L)) \\ &\quad + \frac{1}{\Delta x} \int_0^{\Delta t} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} s \left(\mathcal{W}_R \left(\frac{x}{t}, \tilde{w}_L, \tilde{w}_R \right), z(x) \right) dx dt.\end{aligned}$$

Weak consistency condition :

$$\frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \widehat{\mathcal{W}}_R \left(\frac{x}{\Delta t}, \tilde{w}_L, \tilde{w}_R \right) dx = \frac{w_L + w_R}{2} - \frac{\Delta t}{\Delta x} (f(w_R) - f(w_L)) + \frac{\Delta t}{\Delta x} S(\tilde{w}_L, \tilde{w}_R),$$

where $S(\tilde{w}_L, \tilde{w}_R)$ is such that

$$S((w, z_L), (w, z_R)) = s_{cor}(w) \Delta x + s_{topo}(w)[z],$$

for the continuous source term $s(\tilde{w}) = s_{cor}(w) + s_{topo}(w) \partial_x z$.