Uniform global asymptotic synchronization of a network of Kuramoto oscillators via hybrid coupling

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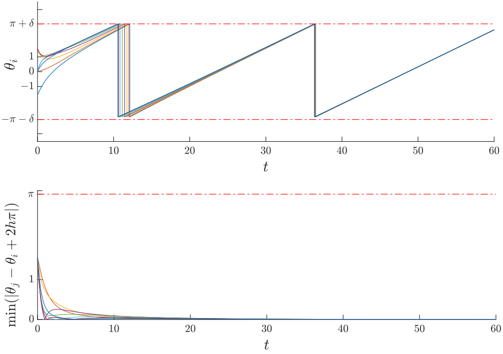
Kuramoto model to describe synchronization phenomena

To model a network of interconnected oscillators:



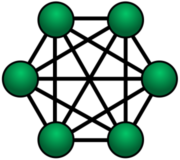
- in chemistry as for flame dynamics [Forrester, Sci. Rep. 2015]
- in biology and neurosciences [Cumin & Unsworth, Physica D. 2007]
- etc.

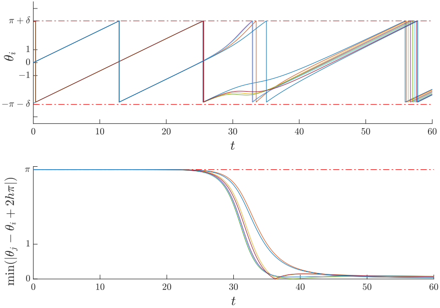
Problematic behaviour in historical Kuramoto

- **Fully connected** network of n oscillators
- Phase dynamics

$$\dot{\theta}_i = \omega + \gamma \sum_{j \in \mathcal{V}} \sin(\theta_j - \theta_i),$$

where $\theta_i \in \mathbb{R}$, $\gamma \in \mathbb{R}_{>0}$ and $\omega \in \mathbb{R}$

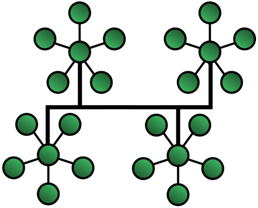




- Various synchronization results [Strogatz, Physica D. 2000], [Dörfler & Bullo, SIAM 2011]
- **Non-uniform synchronization**

Contributions

- **Tree-like network** of *n* oscillators
- Exploiting **hybrid tools** to modify the historical Kuramoto model
- Synchronization via **hybrid coupling**
- **Uniformly globally asymptotically stable** synchronization set



- Absence of undesired non-uniform behavior with large synchronization errors
- Same behavior as the historical Kuramoto with small synchronization errors

Overview

1. Introduction
2. Hybrid model
3. Fundamental properties
4. Numerical examples
5. Conclusions

Overview

1 Introduction

2 Hybrid model

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Historical Kuramoto model

- **Fully connected** network of n oscillators, undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}_u)$
- Phase dynamics

$$\dot{\theta}_i = \omega + \gamma \sum_{j \in \mathcal{V}} \sin(\theta_j - \theta_i)$$

- $\theta_i \in \mathbb{R}$ **i-th oscillator phase**, $i \in \mathcal{V}$
- Natural frequency $\omega \in \mathbb{R}$
- Gain $\gamma \in \mathbb{R}_{>0}$ associated to intensity of coupling action

Hybrid Kuramoto: continuous dynamics

- **Tree-like** network of n oscillators, undirected graph $\mathcal{G}_u = (\mathcal{V}, \mathcal{E}_u)$
- Phase dynamics for each $i \in \mathcal{V}$

$$\dot{\theta}_i = \omega + \gamma \sum_{j \in \mathcal{V}_i} \sigma(\theta_j - \theta_i + 2k_{ij}\pi), \qquad (\theta, k) \in C$$

- $\theta_i \in [-\pi-\delta, \pi+\delta]$, $\delta \in (0, \pi)$ regularisation parameter
- $k_{ij} = -k_{ji} \in \{-1, 0, 1\}$ **unwinding variable**, $(i, j) \in \mathcal{E}_u$

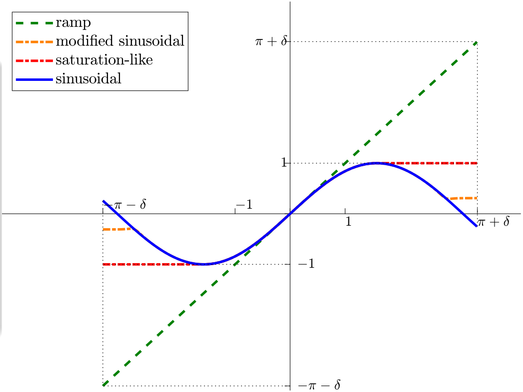
$$\dot{k}_{ij} = 0, \qquad (\theta, k) \in C$$

Function σ to overcome the limits of the standard coupling

Recall $\dot{\theta}_i = \omega + \gamma \sum_{j \in \mathcal{V}_i} \sigma(\theta_j - \theta_i + 2k_{ij}\pi), \quad (\theta, k) \in C$

Property 1

- σ defined on $[-\pi - \delta, \pi + \delta]$
- σ is **odd**
- $\sigma(s)s > 0, \forall s \in [-\pi - \delta, \pi + \delta] \setminus \{0\}$ and $\sigma(0) = 0$



Hybrid Kuramoto: continuous dynamics

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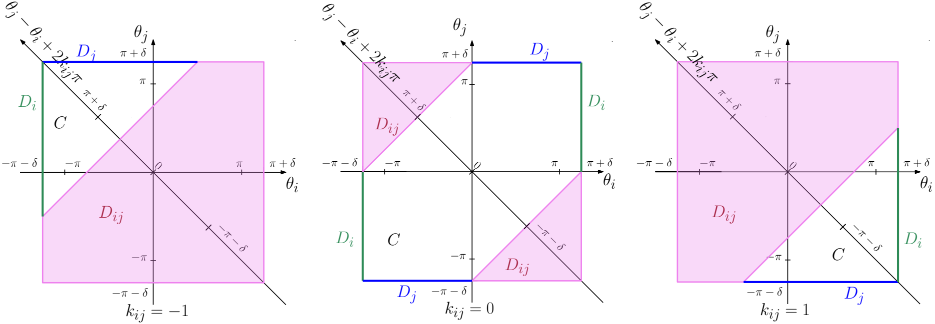
Hybrid Kuramoto: hybrid dynamics and state space

- Tree-like network, $\mathcal{G}_u = (\mathcal{V}, \mathcal{E}_u)$, $|\mathcal{V}| = n$ and $|\mathcal{E}_u| = m$
- $\theta_i \in [-\pi - \delta, \pi + \delta]$, $i \in \mathcal{V}$; $k_{ij} = -k_{ji} \in \{-1, 0, 1\}$, $(i, j) \in \mathcal{E}_u$
- Hybrid dynamics[1] of $(\theta, k) \in X := [-\pi - \delta, \pi + \delta]^n \times \{-1, 0, 1\}^m$

$$\begin{cases} (\dot{\theta}, \dot{k}) = f(\theta, k), & (\theta, k) \in C \\ (\theta^+, k^+) \in G(\theta, k), & (\theta, k) \in D \end{cases}$$



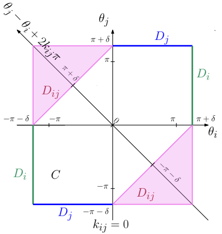
1, Goebel, Teel, Sanfelice, Princeton University Press 2012

Flowing is only allowed in the domain of σ

Recall $\dot{\theta}_i = \omega + \gamma \sum_{j\in\mathcal{V}_i} \sigma(\theta_j - \theta_i + 2k_{ij}\pi), \quad (\theta, k) \in C$

- σ defined on $[-\pi-\delta, \pi+\delta] \Rightarrow$ flow needs to happen only if $|\theta_j - \theta_i + 2k_{ij}\pi| \le \pi + \delta$, $(i,j) \in \mathcal{E}_u$
- Allow jumping from the closed complement: $D_{ij} := \{(\theta, k) \in X : |\theta_j - \theta_i + 2k_{ij}\pi| \ge \pi + \delta\}$

$$\begin{aligned} \theta_i^+ &= \theta_i, \\ \theta_j^+ &= \theta_j, \\ k_{ij}^+ &\in \operatorname*{argmin}_{h\in\{-1,0,1\}} |\theta_j - \theta_i + 2h\pi|, \end{aligned} \qquad (\theta, k) \in D_{ij}$$



Other states remain unchanged (distributed update law)

Lemma

For any $(i,j) \in \mathcal{E}_u$ and $(\theta, k) \in D_{ij}$, (θ^+, k^+) above implies $(\theta^+, k^+) \in X$ and $|\theta_j^+ - \theta_i^+ + 2k_{ij}^+\pi| < \pi + \delta$

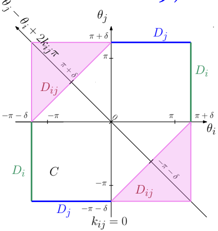
Jump dynamics to unwind phases

Recall $\dot{\theta}_i = \omega + \gamma \sum_{j\in\mathcal{V}_i} \sigma(\theta_j - \theta_i + 2k_{ij}\pi), \quad (\theta, k) \in C$

- $\theta_i \in [-\pi - \delta, \pi + \delta] \Rightarrow$ need for a jump rule when $|\theta_i| = \pi + \delta$
- Allow jumping from
  $D_i := \mathrm{cl}\big(\{(\theta, k) \in X : (\theta, k) \notin D_{ij}$ for any $j \in \mathcal{V}_i$ and $|\theta_i| = \pi + \delta\}\big)$

$$\theta_i^+ = \begin{cases} \theta_i - 2\pi, & \text{if } \theta_i = \pi + \delta, \\ \theta_i + 2\pi, & \text{if } \theta_i = -\pi - \delta, \end{cases} \qquad (\theta, k) \in D_i$$

$$k_{ij}^+ = k_{ij} - \mathrm{sgn}(\theta_i),\ j \in \mathcal{V}_i,$$



Other states remain unchanged (distributed update law)

Lemma

For each $i \in \mathcal{V}$ and $x \in D_i$, (θ^+, k^+) above satisfies implies $(\theta^+, k^+) \in X$, $|\theta_i^+| < \pi + \delta$ and $\theta_j^+ - \theta_i^+ + 2k_{ij}^+\pi = \theta_j - \theta_i + 2k_{ij}\pi$ for each $j \in \mathcal{V}_i$

Hybrid Kuramoto: hybrid dynamics and state space

- Tree-like network, $\mathcal{G}_u = (\mathcal{V}, \mathcal{E}_u)$, $|\mathcal{V}| = n$ and $|\mathcal{E}_u| = m$
- $\theta_i \in [-\pi - \delta, \pi + \delta]$, $i \in \mathcal{V}$; $k_{ij} = -k_{ji} \in \{-1, 0, 1\}$, $(i, j) \in \mathcal{E}_u$
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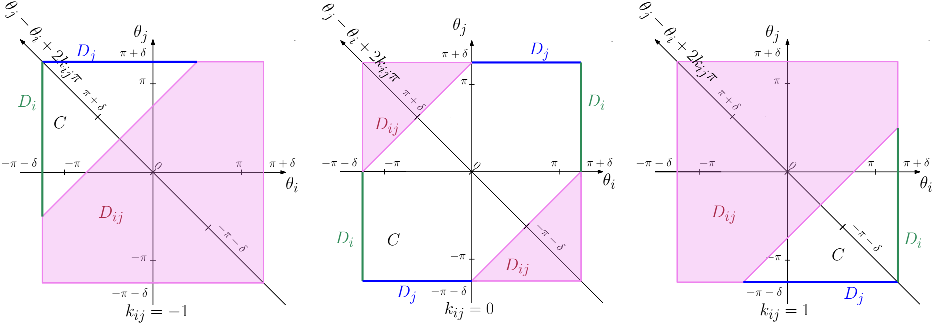

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Hybrid Kuramoto model - UGAS and t-completeness

Set $\mathcal{A}$ as the **synchronization set**

$$\mathcal{A} := \{(\theta, k) : \theta_i = \theta_j + 2k_{ij}\pi,\ (i,j) \in \mathcal{E}_u\}$$

Tree-like network $\Rightarrow$ all phases are equal modulo 2π

Theorem 1

Under Property 1 and with a tree graph, for any $\gamma > 0$

- All maximal solutions ϕ are ***t*-complete**, i.e. any maximal solution ϕ is such that
  $\sup_t \operatorname{dom} \phi = \sup\{t \in \mathbb{R}_{\geq 0} : \exists j \in \mathbb{Z}_{\geq 0} | (t, j) \in \operatorname{dom} \phi\} = +\infty$
- The set $\mathcal{A}$ is **UGAS**, equivalently there exists $\beta \in \mathcal{KL}$ such that $|\phi(t,j)|_{\mathcal{A}} \leq \beta(|\phi(0,0)|_{\mathcal{A}}, t+j)$ for all $(t,j) \in \operatorname{dom} \phi$ and where $|x|_{\mathcal{A}}$ be the distance of $x \in \mathbb{R}^n$ to the closed set $\mathcal{A} \subset \mathbb{R}^n$

Proof is based on a weak Lyapunov function

- Weak Lyapunov function

$$V(\theta, k) := \sum_{(i,j)\in\mathcal{E}} \int_0^{\theta_j - \theta_i + 2k_{ij}\pi} \sigma(\mathrm{sat}_{\pi+\delta}(s))ds, \qquad \forall x \in X$$

- There exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ independent of $\gamma > 0$ such that

$$\text{(S)} \qquad \alpha_1(|(\theta, k)|_{\mathcal{A}}) \leq V(\theta, k) \leq \alpha_2(|(\theta, k)|_{\mathcal{A}}), \qquad \forall(\theta, k) \in X$$

$$\text{(F)} \qquad \langle \nabla V(\theta, k), f(\theta, k)\rangle \leq -\frac{1}{2}\gamma\alpha_3(V(\theta, k)), \qquad \forall(\theta, k) \in C$$

$$\text{(J)} \qquad V(g) - V(\theta, k) \leq 0, \qquad \forall g \in G(\theta, k), \quad \forall(\theta, k) \in D$$

- UGAS of $\mathcal{A}$ from [Thm. 1, Seuret et al., TAC 2018] (invariance principle) due to the absence of complete discrete solutions

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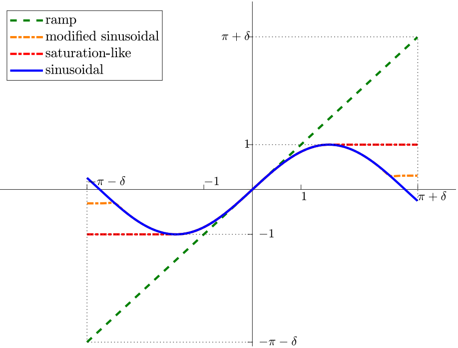
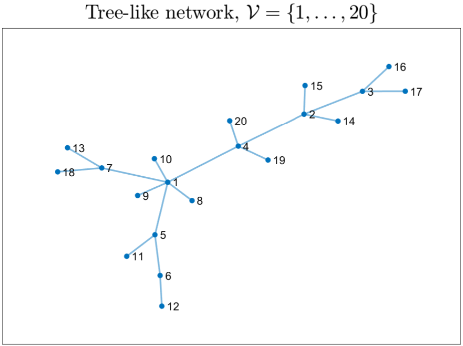
Simulations for different selections of σ and γ

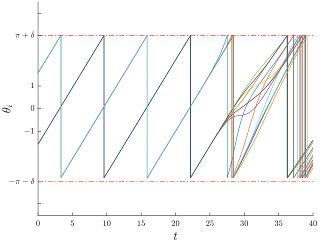
- $n = 20$ agents, $\omega = 0.5$, $\delta = \frac{\pi}{36}$ and $\gamma \in \{0.15, 0.6, 3\}$
- Same γ for all the edges
- σ: sinusoidal, ramp, modified sinusoidal and saturation-like
- Initial conditions close to phase opposition



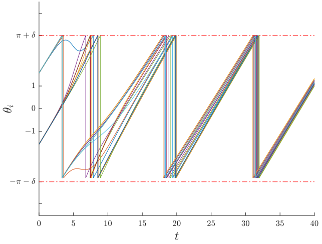


Sinusoid failing Property 1 vs σ's satisfying Property 1

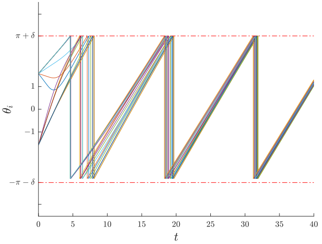
coupling action: sinusoid | γ = **0.15**



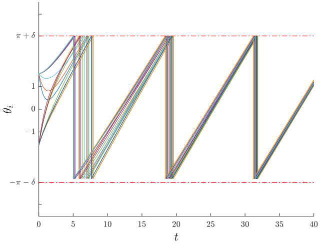
coupling action: modified sinusoid | γ = **0.15**



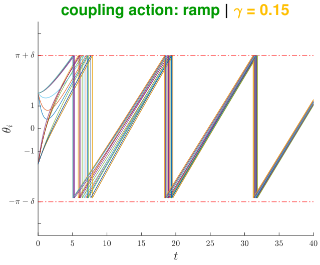
coupling action: saturation-like | γ = **0.15**

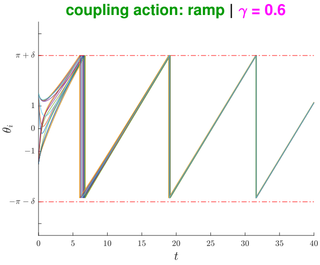


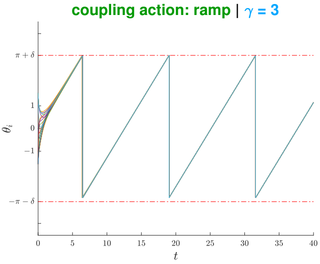
coupling action: ramp | γ = **0.15**

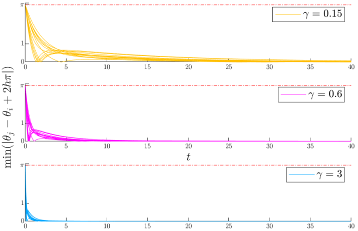


Larger γ's induce faster synchronization



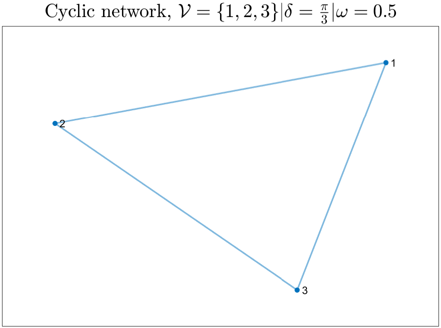


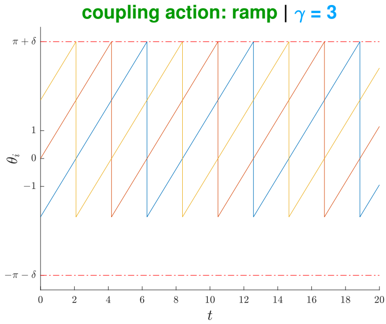




Example of phase lock in case of fully connected graph

The graph is not a tree: Theorem 1 does not apply





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Conclusion and perspectives

Summary:

- A hybrid model of Kuramoto oscillators connected via hybrid coupling
- Uniform stability and attractivity properties of the synchronization set are guaranteed
- Synchronization reached via a leaderless, distributed approach

Future developments:

- Non-identical oscillators with time-varying natural frequencies
- Different topologies and hybrid coupling dynamics

Preliminary work [Bertollo et al., IFAC 2020]
Journal paper under preparation