Quantitative convergence of stochastic particle systems with singular interacting potential

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Nonlinear Fokker-Planck equation with singular interaction

- We study on $[0, T) \times \mathbb{R}^d$

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x) - \nabla \cdot \big(u(t,x)\, (K *_x u(t,x)) \big), \\ u(0,x) = u_0(x), \end{cases} \tag{NLFP}$$

  K - **locally integrable** kernel, with singular behaviour at 0, attractive or repulsive.

- Our main interest : stochastic particle approximation of (NLFP).

- Why?
  - Macroscopic to microscopic description (and back!);
  - Numerical schemes...

Classical approach: mean-field interactions

- (NLFP) is seen as the FP equation for the non-linear process

$$\begin{cases} dX_t = \sqrt{2}dW_t + K * u_t(X_t)dt, \\ \mathcal{L}(X_t) = u_t. \end{cases} \tag{NLSDE}$$

Associate to it the stochastic particle system in mean-field interaction:

$$dX_t^{i,N} = \sqrt{2}dW_t^{i,N} + \frac{1}{N}\sum_{j=1}^{N} K(X_t^{i,N} - X_t^{j,N}). \tag{PS}$$

(see for eg. the notes of Sznitman)

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- Probabilistic approach to non-linear FP equations with **irregular** interactions such as:
  - Boltzmann, Burgers, Navier-Stokes, Keller-Segel equations, ...

  studied by many authors:
  - Bossy, Calderoni, Cattiaux, Fournier, Hauray, Jabir, Jabin, Jourdain, Méléard, Osada, Pulvirenti, Talay, ...

Another viewpoint: moderate interaction

Motivated by singular attractive kernels for which

- existence of (PS) is unknown,
- existence ok, but convergence unknown

we study **moderately interacting particles**

$$dX_t^{i,N} = \sqrt{2}dW_t^{i,N} + F\left(K * (V^N * \mu_t^N)(X_t^{i,N})\right) dt,$$

where:

- $V^N(x) = N^{d\alpha} V(N^\alpha x), \alpha \in (0,1)$; V - regular density;
- F - smooth **cut-off** chosen depending on the initial condition.

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- Some historic references : Oelschläger ('85), Méléard-Roelly ('87)

$\rightarrow$ A **semigroup approach** was recently developed by Flandoli, Olivera and their collaborators to get uniform (non-quantitative) convergence of $V^N * \mu^N$ towards a mild solution to:

FKPP, 2d Navier-Stokes equations, PDE-ODE system related to aggregation phenomena, parabolic-elliptic Keller-Segel model.

Our main objectives

What are the minimal assumptions on the kernel K and what is a suitable functional framework for (NLFP) so the following holds?

- Convergence of $\{\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}},\ t \in [0, T]\}$ to the solution (NLFP) when $N \to \infty$:
  - which range of α?
  - what is the rate of convergence ?
- Well-posedness of (NLSDE).
- Propagation of chaos towards (NLSDE) (without the cut-off and the mollifier)

What kind of kernels can we treat?

A typical example in dimension $d \geq 2$ is the family of **Riesz kernels**:

$$K_s(x) = \pm \nabla V_s(x)$$

where

$$V_s(x) := \begin{cases} |x|^{-s} & \text{if } s \in (0, d-1) \\ -\log|x| & \text{if } s = 0 \end{cases}, \quad x \in \mathbb{R}^d.$$

Examples:

- Coulomb interactions: K_s, with $s = d - 2$ $(d \geq 3)$;
- 2d Navier-Stokes equation (vorticity): $K(x) = \frac{x^\perp}{|x|^2}$;
- Parabolic-elliptic Keller-Segel model: $K(x) = -\chi \frac{x}{|x|^d}$ (*attractive*...);
- Some attractive-repulsive kernels.

Precise assumptions on K and α

(A^K):

1. $K \in L^{\boldsymbol{p}}(\mathcal{B}_1)$, for some $\boldsymbol{p} \in [1, +\infty]$;
2. $K \in L^{\boldsymbol{q}}(\mathcal{B}_1^c)$, for some $\boldsymbol{q} \in [1, +\infty]$;
3. There exists $\boldsymbol{r} \geq \max(\boldsymbol{p}', \boldsymbol{q}')$, $\zeta \in (0, 1]$ and $C > 0$ such that for any $f \in L^1 \cap L^{\boldsymbol{r}}(\mathbb{R}^d)$, one has

$$\mathcal{N}_\zeta(K * f) \leq C \|f\|_{L^1 \cap L^{\boldsymbol{r}}(\mathbb{R}^d)}.$$

Here $\mathcal{N}_\zeta$ is the Hölder seminorm of parameter $\zeta \in (0, 1]$.

(H_α): The parameters α and $\boldsymbol{r}$ satisfy

$$0 < \alpha < \frac{1}{d + 2d(\frac{1}{2} - \frac{1}{\boldsymbol{r}}) \vee 0}.$$

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Theorem 1

Let $T < T_{max}$ and assume (A^k) and (H_α). Under suitable conditions on the initial conditions (in part. $\boldsymbol{u}_0 \in \boldsymbol{L}^1 \cap \boldsymbol{L}^{\boldsymbol{r}}(\mathbb{R}^{\boldsymbol{d}})$), the sequence $\{u_t^N = V^N * \mu_t^N,\ t \in [0,T]\}_{N\in\mathbb{N}}$ converges, as $N \to \infty$, towards the unique mild solution u on $[0,T]$ of the (NLFP)*, in the following sense: for any $\varepsilon > 0$ and any $m \geq 1$, there exists a constant $C > 0$ such that for all $N \in \mathbb{N}^*$,*

$$\Big\| \|u^N - u\|_{T, L^1\cap L^r(\mathbb{R}^d)} \Big\|_{L^m(\Omega)} \leq \Big\| \sup_{s\in[0,T]} \|e^{s\Delta}(u_0^N - u_0)\|_{L^1\cap L^r(\mathbb{R}^d)} \Big\|_{L^m(\Omega)} + CN^{-\varrho+\varepsilon},$$

where

$$\varrho = \min\left(\alpha\zeta, \frac{1}{2}\left(1 - \alpha(d + d(1-\frac{2}{\boldsymbol{r}}) \vee 0)\right)\right).$$

Some consequences and remarks

- Same rate for the genuine empirical measure of (PS)

$$\left\| \sup_{t\in[0,T]} \|\mu_t^N - u_t\|_0 \right\|_{L^m(\Omega)} \leq C\, N^{-\varrho+\varepsilon},$$

  where $\|\cdot\|_0$ denotes the Kantorovich-Rubinstein metric

- The rate in the previous results holds almost surely.
- Cannot expect here a $\sqrt{N}$ rate of convergence because of the short range interactions. : "best possible" $N^{-\alpha}$.

Applications

- **Coulomb-type kernels** (like Biot-Savart kernel in $d = 2$, the Riesz kernel with $s = d - 2$),
  - the convergence happens for any $\alpha < \frac{1}{2(d-1)}$ $(d = 2 \to \alpha = (\frac{1}{2})^{-}.)$;
  - the best possible rate of convergence is $\varrho = \left(\frac{1}{2(d+1)}\right)^{-}$ which is obtained for the choice $\alpha = \left(\frac{1}{2(d+1)}\right)^{+}$, $\boldsymbol{r} = +\infty$, $\boldsymbol{\zeta} = 1$.

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- **Keller-Segel parabolic elliptic model** ($d = 2$: global solution $\chi < 8\pi$, blow up in finite time otherwise).
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  - we get the above rate for any value of χ;
  - the result holds even if the PDE explodes in finite time ($\chi > 8\pi$).
- **The Riesz kernels with** $s > d - 2$ do not satisfy Assumption (A^K) - 3. However, by imposing more regularity on the initial conditions and smaller values of α, we get a rate of convergence for singular Riesz kernels with $s \in (d-2, d-1)$.

About the proof

- Derive the SPDE satisfied by the mollified empirical measure u^N in its mild form

$$u_t^N(x) = e^{t\Delta} u_0^N(x) - \int_0^t \nabla \cdot e^{(t-s)\Delta} \langle \mu_s^N, V^N(x - \cdot) F\big(K * u_s^N(\cdot)\big)\rangle \, ds \\ - \frac{1}{N} \sum_{i=1}^{N} \int_0^t e^{(t-s)\Delta} \nabla V^N(x - X_s^{i,N}) \cdot dW_s^i,$$

- For $q \geq 1$ establish that

$$\sup_{N \in \mathbb{N}^*} \sup_{t \in [0,T]} \mathbb{E}\left[\left\| u_t^N \right\|_{L^r(\mathbb{R}^d)}^q \right] < \infty.$$

- Decompose $\|u_t^N - u_t\|_{L^1 \cap L^r}$ in several terms and control these terms thanks to our hypothesis.

Main issue : control the moments of

$$\sup_{t\leq T} \| \frac{1}{N} \sum_{i=1}^{N} \int_0^t e^{(t-s)\triangle} \nabla V^N(X_s^i - \cdot) dW_s^i \|_{L^1 \cap L^r(\mathbb{R}^d)}$$

- not a martingale, fix the time in the heat operator and it becomes one
- to control its $L^1 \cap L^r(\mathbb{R}^d)$ norm in space, use stochastic integration techniques in infinite-dimensional spaces (van Nerveen et al, '07)
- you need the Garsia-Rodemich-Rumsey's lemma to put the sup inside (you loose a bit of the speed of convergence)

Note that this is where the main limitation on α, Assumption (H_α), arises.

The non-linear process and the propagation of chaos

Proposition 1

Let $T < T_{max}$. Assume that u_0 is a probability density function belonging to $L^{\mathbf{r}}(\mathbb{R}^d)$ and that the kernel K satisfies (H_K). Then, the martingale problem related to (NLSDE) *is well-posed.*

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Example: 2-d Keller-Segel parabolic-elliptic equation $\rightarrow$ we obtain the (local) existence of the (NLSDE) for all the values of the sensitivity parameter χ and the propagation of chaos towards it.

Some next steps

1. Numerical applications : use our result to quantify the convergence of a scheme coming from the moderately interacting particles.
2. Remove the cut-off in the definition of the moderately interacting particles.
3. Treat non-Markovian particle systems : like the ones coming from parabolic-parabolic Keller Segel model.