Homogenization of Poisson equation and Stokes system in a class of non-periodically perforated domains

Sylvain Wolf
PhD student under the supervision of Prof. Xavier Blanc
wolf@math.univ-paris-diderot.fr

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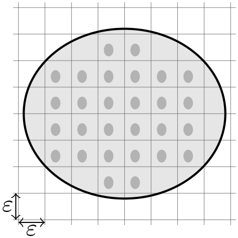
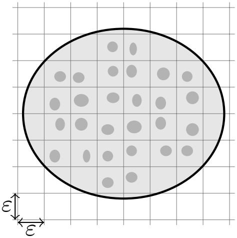
Introduction

- We study the homogenization of Poisson equation and Stokes system in a perforated domain Ω_ε in which
  - the holes scale like ε.
  - two neighbouring holes are separated from a distance $\varepsilon \ll 1$;
- The PDEs we aim at studying are

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in} \quad \Omega_\varepsilon \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega_\varepsilon \end{cases} \quad \text{and} \quad \begin{cases} -\Delta U_\varepsilon + \nabla P_\varepsilon = F \text{ in } \Omega_\varepsilon \\ \text{div } U_\varepsilon = 0 \text{ in } \Omega_\varepsilon \\ U_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon, \end{cases}$$

where the source terms $f \in L^2(\Omega)$ and $F \in L^2(\Omega)^d$.





- Questions:

  1. What is $\lim_{\varepsilon \to 0} u_\varepsilon$ / $\lim_{\varepsilon \to 0} (P_\varepsilon, U_\varepsilon)$?
  2. Can we find *via* a corrector technique an approximation v_ε of u_ε ?
  3. Do we have a convergence Theorem of the form $\|u_\varepsilon - v_\varepsilon\|_{W^{m,p}} \leq C\varepsilon^\beta$?

- The behaviour of u_ε and $(U_\varepsilon, P_\varepsilon)$ is well-known when the holes are periodically distributed in space. We want to extend these results to local perturbations of a given periodic distribution of holes.
  $\longrightarrow$ to model some defects that could appear in the microstructure.

Outline for the talk

1. The setting : periodically and non-periodically perforated domains
2. The Poisson equation
3. The Stokes system

The periodic setting

[Lions J.-L., *Asymptotic expansions in perforated media with a periodic structure*, 1980]

We denote $Q :=]0,1[^d$ and $Q_k := Q + k$, $k \in \mathbb{Z}^d$. Let $\mathcal{O}_0^{\mathrm{per}} \subset\subset Q$ be a $C^{1,\gamma}$ open set s.t. $Q \setminus \overline{\mathcal{O}_0^{\mathrm{per}}}$ is connected and $\mathcal{O}_k^{\mathrm{per}} := \mathcal{O}_0^{\mathrm{per}} + k \subset\subset Q_k, k \in \mathbb{Z}^d$.
Set of perforations : $\mathcal{O}^{\mathrm{per}} := \bigcup_{k \in \mathbb{Z}^d} \mathcal{O}_k^{\mathrm{per}}$.
Let $\Omega \subset \mathbb{R}^d$ be a bounded domain (macroscopic domain).
Perforated domain : $\Omega_\varepsilon^{\mathrm{per}} = \Omega \setminus \bigcup_{k \in Y_\varepsilon} \varepsilon \mathcal{O}_k^{\mathrm{per}}$ where $Y_\varepsilon := \{k \in \mathbb{Z}^d,\ \varepsilon Q_k \subset \Omega\}$.
The set $\Omega_\varepsilon^{\mathrm{per}}$ is open, bounded, connected and has the same regularity as Ω and $\mathcal{O}^{\mathrm{per}}$.

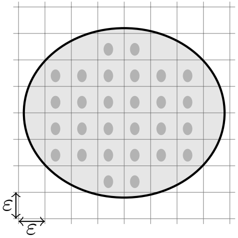


Figure: The domain $\Omega_\varepsilon^{\mathrm{per}}$

The non-periodic setting

We fix a sequence $(\mathcal{O}_k^{\mathrm{per}})_{k\in\mathbb{Z}^d}$ as before. We define the non-periodic perforations $(\mathcal{O}_k)_{k\in\mathbb{Z}^d}$ by the following properties:

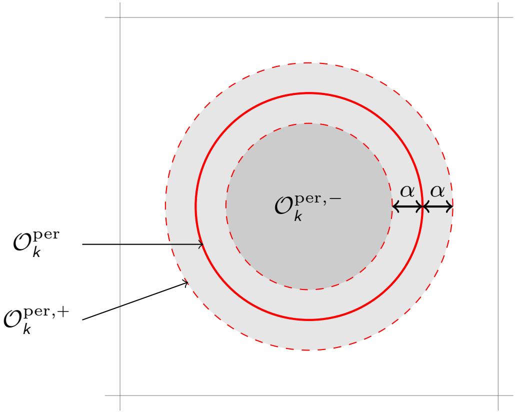
- **(A1)** For all $k \in \mathbb{Z}^d$, $\mathcal{O}_k \subset\subset Q_k$ and $Q_k \setminus \overline{\mathcal{O}_k}$ is connected;
- **(A2)** Geometric hypothesis :

$$\exists (\alpha_k)_{k\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d) \text{ s.t } \mathcal{O}_k^{\mathrm{per},-}(\alpha_k) \subset \mathcal{O}_k \subset \mathcal{O}_k^{\mathrm{per},+}(\alpha_k)$$

  where for $\alpha > 0$, $\mathcal{O}_k^{\mathrm{per},+}(\alpha) = \{x \in \mathbb{R}^d \text{ s.t } \mathrm{dist}(x, \mathcal{O}_k^{\mathrm{per}}) < \alpha\}$ and $\mathcal{O}_k^{\mathrm{per},-}(\alpha) = \{x \in \mathcal{O}_k^{\mathrm{per}} \text{ s.t } \mathrm{dist}(x, \partial\mathcal{O}_k^{\mathrm{per}}) > \alpha\}$.

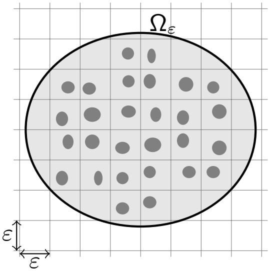
- **Regularity assumption.** There exist $r > 0$, $\alpha > 0$ and $M > 0$ such that for all $k \in \mathbb{Z}^d$ and $x^k \in \partial\mathcal{O}_k$, there exists a flattening chart ψ_{x^k} of class $\mathcal{C}^{1,\alpha}(B_r(x^k))$ such that $\|\Psi_{x^k}\|_{\mathcal{C}^{1,\alpha}(B_r(x^k))} \leq M$.

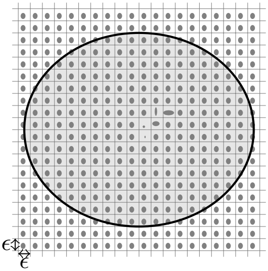
Illustration of (A2)



The non-periodically perforated domain

We define $\mathcal{O} := \bigcup_{k\in\mathbb{Z}^d} \mathcal{O}_k$ and $\Omega_\varepsilon := \Omega \setminus \bigcup_{k\in Y_\varepsilon} \varepsilon\mathcal{O}_k$ where $Y_\varepsilon := \{k \in \mathbb{Z}^d, \quad \varepsilon Q_k \subset \Omega\}$.





Examples:

1) Compactly supported perturbations

2) ℓ^1-translations of the periodic case

Poisson equation

Two-scale expansion:

$$u_\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^3 u_3\left(x, \frac{x}{\varepsilon}\right) + \cdots, \qquad (1)$$

where $u_i : \Omega \times (\mathbb{R}^d \backslash \overline{\mathcal{O}}) \to \mathbb{R}$ and we find the u_i's. We write

$$-\Delta u_\epsilon = f \quad \underset{y=\frac{x}{\epsilon}}{\Longrightarrow} \quad \begin{cases} -\Delta_y u_0 = 0 \\ -\Delta_y u_1 - 2(\nabla_x \cdot \nabla_y) u_0 = 0 \\ -\Delta_y u_2 - 2(\nabla_x \cdot \nabla_y) u_1 - \Delta_x u_0 = f \\ \ldots \end{cases}$$

and we solve the equations. This gives

$$u_\varepsilon(x) = \cancel{u_0\left(x, \frac{x}{\varepsilon}\right)} + \cancel{\varepsilon u_1\left(x, \frac{x}{\varepsilon}\right)} + \varepsilon^2 w\left(\frac{x}{\varepsilon}\right) f(x) + \varepsilon^3 u_3\left(x, \frac{x}{\varepsilon}\right) + \cdots.$$

where the corrector function w solves the PDE $\begin{cases} -\Delta w = 1 \text{ in } \mathbb{R}^d \setminus \overline{\mathcal{O}} \\ w = 0 \text{ on } \partial\mathcal{O}. \end{cases}$

In the periodic case *i.e* when $\mathcal{O} = \mathcal{O}^{\mathrm{per}}$, the corrector PDE is readily solvable and gives a solution w^{per} such that $w^{\mathrm{per}} \in H^{1,\mathrm{per}}(Q \setminus \overline{\mathcal{O}^{\mathrm{per}}})$.

The non-periodic corrector

Theorem

*Suppose that **(A1)-(A2)** are satisfied. The PDE*

$$\begin{cases} -\Delta w = 1_{\mathbb{R}^d \setminus \overline{\mathcal{O}^{\mathrm{per}}}} + \widetilde{g} \\ w_{|\partial\mathcal{O}} = 0 \end{cases}$$

where $\widetilde{g} \in L^2(\mathbb{R}^d)$ admits a unique solution of the form $w = w^{\mathrm{per}} + \widetilde{w}$ where w^{per} is the periodic corrector and $\widetilde{w} \in H^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$.

Strategy of the proof. The PDE on $\widetilde{w}$ reads:

$$-\Delta \widetilde{w} = 1_{\mathbb{R}^d \setminus \overline{\mathcal{O}^{\mathrm{per}}}} + \widetilde{g} + \Delta w^{\mathrm{per}}.$$

We apply standard Lax-Milgram Lemma in $H^1_0(\mathbb{R}^d \setminus \overline{\mathcal{O}})$:
→ there exists a function $\phi \in H^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$ such that $\phi = -w^{\mathrm{per}}$ on $\partial\mathcal{O}$;
→ the RHS satisfies $1_{\mathbb{R}^d \setminus \overline{\mathcal{O}^{\mathrm{per}}}} + \widetilde{g} + \Delta w^{\mathrm{per}} \in H^{-1}(\mathbb{R}^d \setminus \overline{\mathcal{O}})$;
→ there holds a Poincaré inequality.

Convergence rates

Theorem

Let $f \in \mathcal{D}(\Omega)$. There exists a constant $C > 0$ independent of ε such that

$$\|u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f\|_{H_0^1(\Omega_\varepsilon)} \leq C\varepsilon^2$$

We set $R_\varepsilon := u_\varepsilon - \varepsilon^2 w(\cdot/\varepsilon) f$ and we compute:

$$-\Delta R_\varepsilon = \varepsilon g_\varepsilon, \quad g_\varepsilon := 2\nabla w\left(\frac{\cdot}{\varepsilon}\right) \cdot \nabla f + \varepsilon w\left(\frac{\cdot}{\varepsilon}\right) \Delta f$$

It is easily proved that $\|g_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C$. Thus, by integrating by parts and using Poincaré inequality in Ω_ε in form of[1]

$$\forall v \in H_0^1(\Omega_\varepsilon), \quad \|v\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon \|\nabla v\|_{L^2(\Omega_\varepsilon)},$$

we conclude the proof.

NB. We can weaken the hypothesis to $f \in H^2(\Omega)$ and $f_{|\partial\Omega} = 0$ provided regularity on the perforations. If $f_{|\partial\Omega} \neq 0$, then $\|R_\varepsilon\|_{H_0^1(\Omega_\varepsilon)} \leq C\varepsilon^{3/2}$.

[1] See [Blanc, W., *Homogenization of Poisson equation in a non periodically perforated domain*, 2021] or [Donato, Picard, *Convergence for Dirichlet problems for monotone operators in a class of porous media*, 2004].

Comments

1) We can obtain higher order approximation results of the form:

$$\left\| u_\varepsilon - \varepsilon^2 w\left(\frac{\cdot}{\varepsilon}\right) f - \varepsilon^3 z^j \left(\frac{\cdot}{\varepsilon}\right) \partial_j f \right\|_{H_0^1(\Omega_\varepsilon)} \leq C\varepsilon^3.$$

2) Microscopic perturbation $\Longrightarrow$ microscopic effect ;
a) The (macroscopic) L^2-weakly convergence is $u_\varepsilon/\varepsilon^2 \underset{\varepsilon\to 0}{\longrightarrow} \left(\int_Q w^{\mathrm{per}}(y)\mathrm{d}y\right) f$, the same as the periodic case.
b) This also means that we need fine convergence Theorems to prove that the use of the non-periodic corrector in fact improves the rate of convergence.

We have:

$$\|\varepsilon^2 \widetilde{w}(./\varepsilon) f\|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^{1+d/2},$$

thus, if $d \geq 2$, we get that

$$\|u_\varepsilon - \varepsilon^2 w^{\mathrm{per}}(./\varepsilon) f\|_{H^1(\Omega_\varepsilon)} \leq C\varepsilon^2.$$

$\rightarrow$ same rate of convergence!

Improved rates of convergence for Poisson equation

See [Masmoudi, *Some uniform elliptic estimates in porous medium*, 2004] in the periodic case.

Theorem

Let $f \in \mathcal{D}(\Omega)$. Suppose the regularity assumption. For all $1 < p < +\infty$, there exists a constant $C > 0$ such that

$$\|u_\varepsilon - \varepsilon^2 w(./\varepsilon) f\|_{W^{1,p}(\Omega_\varepsilon)} \leq C\varepsilon^2.$$

There exists a constant $C > 0$ such that

$$\|u_\varepsilon - \varepsilon^2 w(./\varepsilon) f\|_{L^\infty(\Omega_\varepsilon)} \leq C\varepsilon^3.$$

Remarks.

1. We can prove $W^{m,q}(\Omega_\varepsilon)-$estimates.
2. If $f \neq 0$ on $\partial\Omega$, we have the estimate: $\|u_\varepsilon - \varepsilon^2 w(./\varepsilon) f\|_{W^{1,p}(\Omega_\varepsilon)} \leq C\varepsilon^{1+\frac{1}{p}}$.
3. Since $\|\varepsilon^2 \widetilde{w}(./\varepsilon) f\|_{W^{1,p}(\Omega_\varepsilon)} \leq C\varepsilon^{1+\frac{d}{p}}$, we have that $\|u_\varepsilon - \varepsilon^2 w^{\mathrm{per}}(./\varepsilon) f\|_{W^{1,p}(\Omega_\varepsilon)} \leq C\varepsilon^{1+\frac{d}{p}}$: the $W^{1,p}$, $p > d$ and L^∞-convergence are less accurate if we use w^{per} instead of w.

Formal homogenization of the Stokes system

Brief review of the periodic case [Tartar, Sanchez-Palencia '77]. We suppose that the following two-scale expansion is satisfied:

$$\begin{cases} U_\varepsilon(x) = u_0(x, x/\varepsilon) + \varepsilon u_1(x, x/\varepsilon) + \varepsilon^2 u_2(x, x/\varepsilon) + \cdots \\ P_\varepsilon(x) = p_0(x, x/\varepsilon) + \varepsilon p_1(x, x/\varepsilon) + \varepsilon^2 p_2(x, x/\varepsilon) + \cdots , \end{cases}$$

where $u_i : \Omega \times (Q \setminus \overline{\mathcal{O}}^{\mathrm{per}}) \to \mathbb{R}^d$ and $p_i : \Omega \times (Q \setminus \overline{\mathcal{O}^{\mathrm{per}}}) \to \mathbb{R}$ are periodic in the second variable. We find that, formally,

$$U_\varepsilon \simeq \varepsilon^2 \sum_{j=1}^{d} w_j(./\varepsilon)(f_j - \partial_j p_0) \quad \text{and} \quad P_\varepsilon(x) \simeq p_0 + \varepsilon \sum_{j=1}^{d} p_j(./\varepsilon)(f_j - \partial_j p_0),$$

where p_0 satisfies the Darcy's law:

$$\begin{cases} -\mathrm{div} A(f - \nabla p_0) = 0 & \text{in} \quad \Omega \\ A(f - \nabla p_0) \cdot n = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{2}$$

and the corrector functions $(w_j^{\mathrm{per}}, p_j^{\mathrm{per}})$ satisfy

$$\begin{cases} -\Delta w_j^{\mathrm{per}} + \nabla p_j^{\mathrm{per}} = e_j & \text{in} \quad Q \setminus \overline{\mathcal{O}^{\mathrm{per}}} \\ \mathrm{div}(w_j^{\mathrm{per}}) = 0 & \\ w_j^{\mathrm{per}} = 0 & \text{on} \quad \partial\mathcal{O}^{\mathrm{per}}. \end{cases} \tag{3}$$

Existence of the non-periodic correctors

We search w_j under the form $w_j = w_j^{\mathrm{per}} + \widetilde{w}_j$ and $p_j = p_j^{\mathrm{per}} + \widetilde{p}_j$. The PDEs defining $(\widetilde{w}_j, \widetilde{p}_j)$ read

$$\begin{cases} -\Delta \widetilde{w}_j + \nabla \widetilde{p}_j = e_j + \Delta w_j^{\mathrm{per}} - \nabla p_j^{\mathrm{per}} \quad \text{in} \quad \mathbb{R}^d \setminus \overline{\mathcal{O}} \\ \mathrm{div}(\widetilde{w}_j) = 0 \\ \widetilde{w}_j = -w_j^{\mathrm{per}} \quad \text{on} \quad \partial\mathcal{O}. \end{cases} \tag{4}$$

Theorem

There exists a solution $(\widetilde{w}_j, \widetilde{p}_j) \in H_0^1(\mathbb{R}^d \setminus \overline{\mathcal{O}}) \times L^2_{loc}(\mathbb{R}^d \setminus \overline{\mathcal{O}})$ *to* (4). *Besides, the following estimate holds true:* $\|\widetilde{p}_j - \fint_{Q_R} \widetilde{p}_j\|_{L^2(Q_R \setminus \overline{\mathcal{O}})} \le CR$, *where the constant* C *is independent of* R.

Ideas on the proof. We can apply Lax-Milgram Lemma as for Poisson equation. We need the following Lemma.

Lemma

The problem $\begin{cases} div(u) = 0 \\ u_{|\partial\mathcal{O}} = -w_j^{\mathrm{per}} \end{cases}$ *admits a solution* $u \in H^1(\mathbb{R}^d \setminus \overline{\mathcal{O}})$.

Convergence Theorem

Proposition

We have the L^2-strong convergence:
$\varepsilon^{-2}\left(U_\varepsilon - \varepsilon^2 \sum_{j=1}^d w_j(./\varepsilon)(f_j - \partial_j p_0)\right) \underset{\varepsilon\to 0}{\longrightarrow} 0$ *where p_0 is the Darcy's pressure.*

Remark. No change of the macroscopic behaviour compared to the periodic case: $\varepsilon^{-2}u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} A(f - \nabla p_0)$ L^2-weakly.

We aim at providing convergence rates in the case $d = 3$. We make two simplifications:

- We impose conditions on the force field f to avoid boundary effects:

$$\mathrm{div}(Af) = 0 \quad \text{and} \quad f \text{ compactly supported.}$$

$$\Longrightarrow U_\varepsilon - \varepsilon^2 \sum_{j=1}^{3} w_j(./\varepsilon)(f_j - \partial_j p_0) = 0 \quad \text{on} \quad \partial\Omega.$$

- We study convergence rates in H^m-norms.

For the periodic case, see [Shen, *Sharp convergence rates for Darcy's law*, 2020].

Convergence Theorem

Theorem (H^2-estimates)

Let $f \in \left[W^{3,\infty}(\Omega)\right]^3$ s.t. $\operatorname{div}(Af) = 0$ and f is compactly supported in Ω. Under the regularity assumption, there exists a constant $C > 0$ independent of ε s.t.

$$\left\|D^2\left[U_\varepsilon - \varepsilon^2 w_j\left(\frac{\cdot}{\varepsilon}\right)f_j\right]\right\|_{L^2(\Omega_\varepsilon)^3} + \varepsilon^{-1}\left\|\nabla\left[U_\varepsilon - \varepsilon^2 w_j\left(\frac{\cdot}{\varepsilon}\right)f_j\right]\right\|_{L^2(\Omega_\varepsilon)^3} + \varepsilon^{-2}\left\|U_\varepsilon - \varepsilon^2 w_j\left(\frac{\cdot}{\varepsilon}\right)f_j\right\|_{L^2(\Omega_\varepsilon)^3} \leq C\varepsilon$$

and

$$\left\|\nabla\left[P_\varepsilon - \varepsilon\left\{p_j\left(\frac{\cdot}{\varepsilon}\right) - \lambda_\varepsilon^j\right\}f_j\right]\right\|_{L^2(\Omega_\varepsilon)} + \left\|P_\varepsilon - \varepsilon\left\{p_j\left(\frac{\cdot}{\varepsilon}\right) - \fint_{\Omega_\varepsilon} p_j\left(\frac{\cdot}{\varepsilon}\right)\right\}f_j\right\|_{L^2(\Omega_\varepsilon)/\mathbb{R}} \leq C\varepsilon.$$

NB. We can obtain H^m-estimates, $m > 2$.

Summary. Construction of a framework that allow to treat defects in the micro-structure of a given periodically perforated domain;

Study of the Poisson equation (existence of correctors, fine convergence Theorems);

Study of the Stokes system (existence of correctors, convergence Theorems in H^m-norms).

References.

Blanc X., Wolf S., Homogenization of the Poisson equation in a non-periodically perforated domain, to appear in Asymptotic Analysis, 2021.

Wolf S., Homogenization of the Stokes system in a non-periodically perforated domain, submitted, 2021.

Thank you for your attention!