# The Logarithmic Schrödinger-Langevin Equation and Quantum Fluids

Quentin Chauleur

PhD advisors : Rémi Carles & Erwan Faou

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# Logarithmic Schrödinger-Langevin Equation

$$\boxed{i\partial_t\psi + \frac{1}{2}\Delta\psi = \lambda\psi\log(|\psi|^2) + \frac{1}{2i}\mu\psi\log\left(\frac{\psi}{\psi^*}\right),} \qquad (1)$$

for $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^d$.

$\Rightarrow$ physics : $\mu > 0$. Influence of $\lambda > 0$ and $\lambda < 0$ ?

- well-posedness ?
- local/global existence ? long time behavior ?
- stationnary solutions ? stability ?
- numerics ?

1. Schrödinger-Langevin Equation
   - Gaussian solutions and related ODE
   - $\lambda < 0$ case
   - $\lambda > 0$ case

2. Quantum Hydrodynamics Systems
   - Isothermal Euler-Langevin-Korteweg Equation
   - Long-time Behavior
   - Splitting Method

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# Gaussian solutions and related ODE

Plug

$$\psi(t,x) = b(t)e^{-\frac{1}{2}a(t)x^2}$$

into (1), $a, b \in \mathbb{C}$. After calculations, you get

$$\boxed{\ddot{r} = \frac{\alpha_0^2}{r^3} + \frac{2\lambda\alpha_0}{r} - \mu\dot{r},} \tag{2}$$

where

$$a(t) = \frac{\alpha_0}{r(t)^2} - i\frac{\dot{r}(t)}{r(t)},$$

$$|b(t)| = |b_0| \exp\left(\frac{1}{2}\operatorname{Im} A(t)\right),$$

with $A(t) = \int_0^t a(s)ds$, $\alpha_0 = \operatorname{Re} a_0 \geq 0$ and $r \in \mathbb{R}$.

# $\lambda < 0$ case

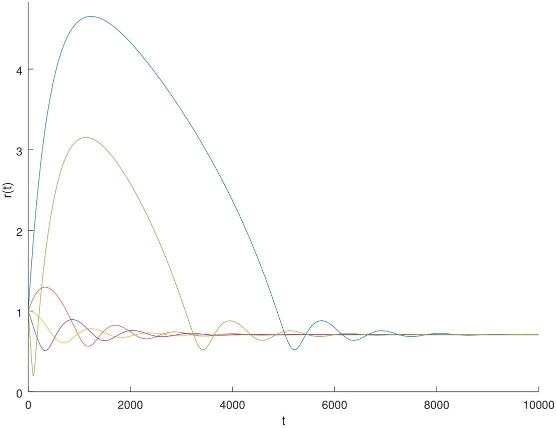


Figure – Dynamic of equation (2)

# $\lambda < 0$ case

## Proposition

$$r(t) \underset{t\to+\infty}{\longrightarrow} \sqrt{\frac{\alpha_0}{-2\lambda}}$$

$$\dot{r}(t) \underset{t\to+\infty}{\longrightarrow} 0$$

## Corollary

*As $a(t) \to -2\lambda$, every Gaussian solution $\psi$ tends to a Gaussian*

$$\phi(x) = C_{\lambda,\alpha_0,\beta_0} e^{\lambda x^2},$$

*uniquely determined by the IC $\alpha_0$ and $b_0$.*

# $\lambda > 0$ case

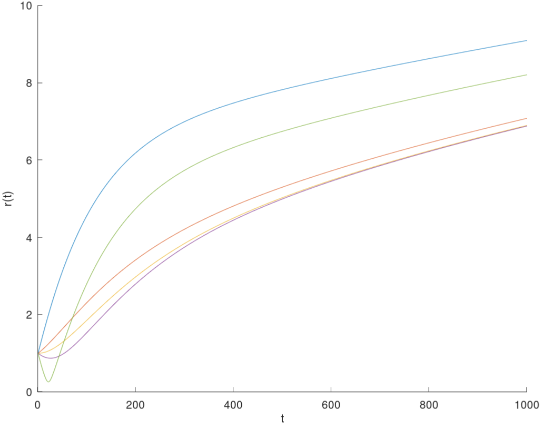


Figure – Dynamic of equation (2)

# $\lambda > 0$ case

## Proposition

$$r(t) \sim 2\sqrt{\frac{\lambda\alpha_0}{\mu}t}$$

$$\dot{r}(t) \sim \sqrt{\frac{\lambda\alpha_0}{\mu t}}$$

## Corollary

$$\|\psi(t)\|_{L^\infty} \sim C_{\mu,\lambda,\alpha_0} t^{-\frac{1}{4}}$$

$$\|\nabla\psi(t)\|_{L^2}^2 \sim C_{\mu,\lambda,\alpha_0,|b_0|} \frac{1}{\sqrt{t}}$$

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# Isothermal Euler-Langevin-Korteweg Equation

We recall :

$$\boxed{i\hbar\partial_t\psi + \frac{\hbar^2}{2}\Delta\psi = \lambda\psi\log(|\psi|^2) + \frac{\hbar}{2i}\mu\psi\log\left(\frac{\psi}{\psi^*}\right)}$$

**Madelung transform** $\psi = \sqrt{\rho}e^{iS/\hbar}$, and define $J = \rho\nabla S/\hbar$, then :

$$\partial_t\rho + \mathrm{div} J = 0 \tag{3}$$

$$\partial_t J + \mathrm{div}\left(\frac{J\otimes J}{\rho}\right) + \lambda\nabla\rho + \mu J = \frac{\hbar^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) \tag{4}$$

**Remark**

Pathological 3rd order quantum potential $\rho\nabla\left(\Delta\sqrt{\rho}/\sqrt{\rho}\right)$.

# Long-time Behavior

**Proposition**

*$\underline{\lambda < 0 \text{ case}}$ : the density $\rho$ of every solution of (3)-(4) converges to a stationary Gaussian $\phi(x) = Ce^{\lambda|x-x_\infty|^2}$ weakly in $L^1(\mathbb{R}^d)$.*

**Proposition**

*$\underline{\lambda > 0 \text{ case}}$ : define*

$$\rho(t,x) = \frac{1}{\tau(t)^d} R\left(t, \frac{x}{\tau(t)}\right)$$

*where $\tau$ is the unique solution of the nonlinear ODE : $\ddot{\tau} = \frac{2\lambda}{\tau} - \mu\dot{\tau}$. Then*

$$R(t,x) \rightharpoonup Ce^{-\frac{|x|^2}{2}} \quad \text{weakly in } L^1(\mathbb{R}^d).$$

# Splitting Method

$$i\partial_t\psi + \frac{1}{2}\Delta\psi = \lambda\psi\log(|\psi|^2) + \frac{1}{2i}\mu\psi\log\left(\frac{\psi}{\psi^*}\right)$$

We solve :

- $\partial_t\psi = -\frac{1}{2}i\Delta\psi$ by FFT,
- $\partial_t\psi = -i\lambda\psi\log(|\psi|^2 + \varepsilon)$ by the explicit solution

$$\psi(t+\Delta t,.) = \psi(t,.)e^{-i\lambda\Delta t\log(|\psi(t,.)|^2+\varepsilon)},$$

- and $\partial_t\psi = -\frac{1}{2}\mu\psi\log\left(\frac{\psi}{\psi^*}\right)$ by an explicit solution

$$\psi(t+\Delta t,.) = a(t,.)e^{i\theta(t,.)e^{-\mu\Delta t}},$$

where we decompose $\psi(t,.) = a(t,.)e^{i\theta(t,.)}$.

# $\lambda < 0$ case

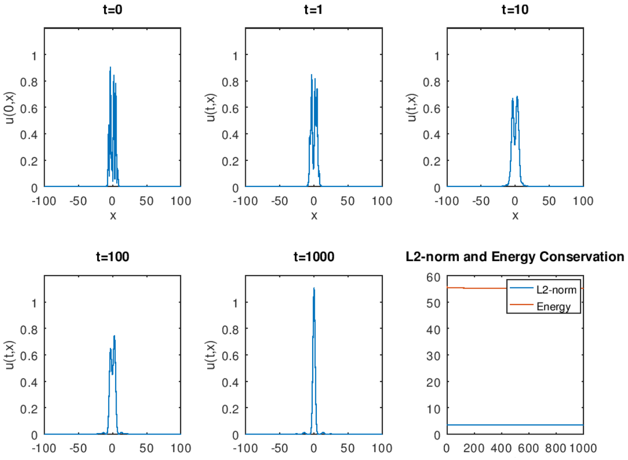


Figure – Solution of equation (1) with initial datum $\psi_0$ in the focusing case ($\lambda = -0.1$, $\mu = 1$).

# $\lambda > 0$ case

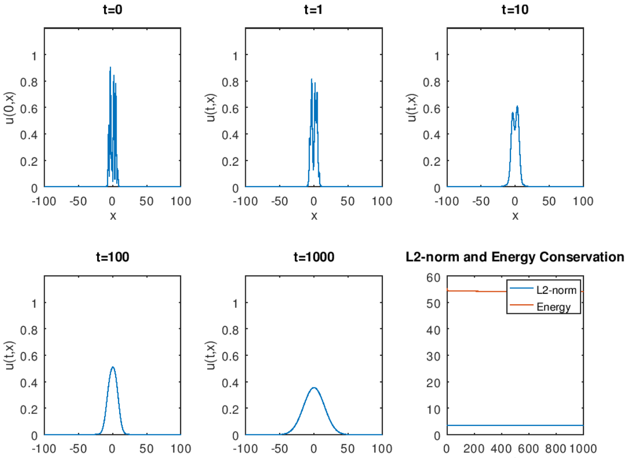


Figure – Solution of equation (1) with initial datum $\psi_0$ in the defocusing case ($\lambda = 0.1$, $\mu = 1$).

# References

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Quentin Chauleur.
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Thank you for your attention.