Reaction-diffusion problems with membrane conditions

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Biological context

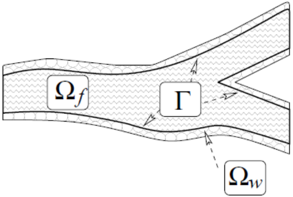


Figure: Arterial wall

- QUARTERONI A., VENEZIANI A., AND ZUNINO P. *SIAM J. Numer. Anal.* (2002).

Biological context

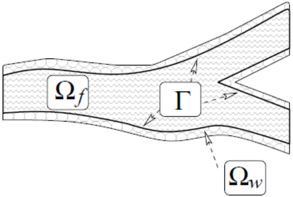


Figure: Arterial wall

Figure: Nuclear pores

- QUARTERONI A., VENEZIANI A., AND ZUNINO P. *SIAM J. Numer. Anal.* (2002).

- SERAFINI A. *Ph.D. Thesis* (2007).
- CANGIANI A. AND NATALINI R. *J. Theor. Biol.* (2010).
- DIMITRIO L. *Ph.D. Thesis* (2012).

Biological context

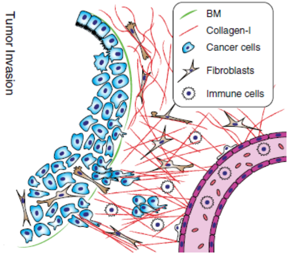
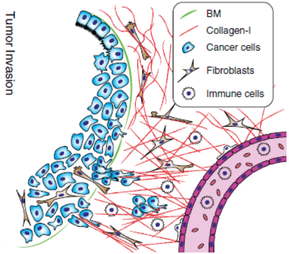


Figure: Tumor invasion

- Gallinato O., Colin T., Saut O., and Poignard C. *J. Theor. Biol.* (2017).
- Chaplain M. A., Giverso C., Lorenzi T., and Preziosi L. *SIAM J. Appl. Math.* (2019).

Biological context



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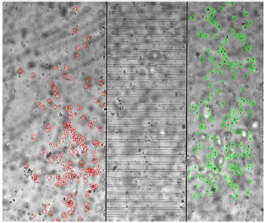
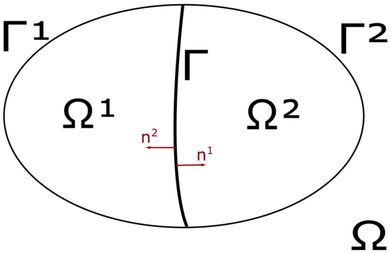


Figure: Organ-on-chip

- GALLINATO O., COLIN T., SAUT O., AND POIGNARD C. *J. Theor. Biol.* (2017).
- CHAPLAIN M. A., GIVERSO C., LORENZI T., AND PREZIOSI L. *SIAM J. Appl. Math.* (2019).

- BRAUN E. C. *PhD Thesis* (2021).
- BRETTI G., DE NINNO A., NATALINI R., PERI D., ROSELLI N. *in preparation* (2021).

Mathematical description



We denote each species density for $i = 1, ..., m$ with

$$u_i = \begin{cases} u_i^1, & \text{in } \Omega^1, \\ u_i^2, & \text{in } \Omega^2, \end{cases}$$

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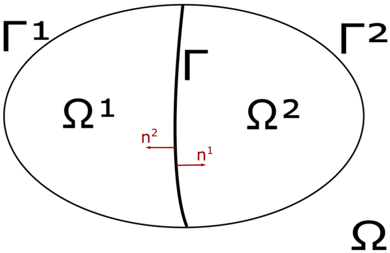

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The reaction-diffusion system for $i = 1, ..., m$:

$$\begin{cases} \partial_t u_i - D_i \Delta u_i = f_i(u_1, ..., u_m), & \text{in } Q_T := (0,T) \times (\Omega^1 \cup \Omega^2), \\ u_i = 0, & \text{in } \Sigma_T := (0,T) \times (\Gamma^1 \cup \Gamma^2), \\ \partial_{\boldsymbol{n}^1} u_i^1 = \partial_{\boldsymbol{n}^1} u_i^2 = k_i(u_i^2 - u_i^1), & \text{in } \Sigma_{T,\Gamma} := (0,T) \times \Gamma, \\ u_i(0,x) = u_{0,i}(x) \geq 0, & \text{in } \Omega, \end{cases}$$

with $D_i > 0$, $k_i \in [0, +\infty]$.

State of art

- **Membrane conditions: Kedem-Katchalsky conditions**
  Kedem O., Katchalsky A. *J. Gen. Physiol.* (1961).

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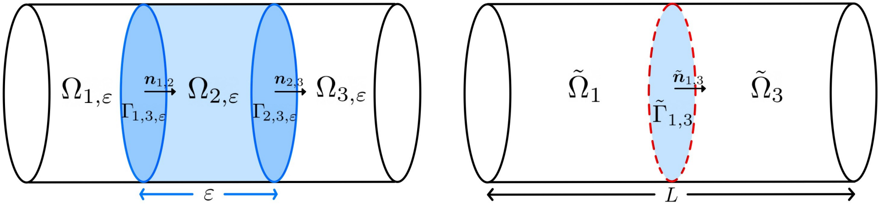

Figure: In the right, domain with a thick membrane and in the left, its limit for the thickness $\varepsilon \to 0$.

- SÁNCHEZ-PALENCIA E. *J. Math. Pures Appl.* (1974).
- C. G., DAVID N., POULAIN A. *preprint* (2021).

State of art

- **Membrane conditions: Kedem-Katchalsky conditions**
  KEDEM O., KATCHALSKY A. *J. Gen. Physiol.* (1961).

- **Global existence in reaction-diffusion systems without membrane conditions**
  - BOTHE D., PIERRE M. *J. Math. Anal. Appl.* (1984).
  - PIERRE M. *Milan J. Math.* (2010).
  - LAAMRI E.-H., PERTHAME B. *J. Differ. Equ.* (2019).

The hypothesis on the reaction terms

$\forall i = 1, ..., m$ and $\forall \boldsymbol{u} = (u_1, ..., u_m) \in [0, +\infty)^m$,

$$|f_i(\boldsymbol{u})| \leq C\Big(1 + \sum_{j=1}^{m} u_j^2\Big), \qquad \text{(sub-quadratic growth)}, \qquad (1)$$

$$\sum_{j=1}^{m} f_j(\boldsymbol{u}) \leq C\Big(1 + \sum_{j=1}^{m} u_j\Big), \qquad \text{(mass control)}, \qquad (2)$$

$$f_i(u_1, ..., u_{i-1}, 0, u_{i+1}, ..., u_m) \geq 0, \qquad \text{(quasi-positivity)}, \qquad (3)$$

$$|f_i(\boldsymbol{u}) - f_i(\boldsymbol{v})| \leq C_M \sum_{j=1}^{m} |u_j - v_j|, \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in [0, M]^m. \qquad (4)$$

Assumption (3) $\Longrightarrow$ u_i are non-negative

Assumption (2) $\Longrightarrow$ mass-control

Main result

Theorem (Existence and regularity)

Assume (1)–(4) *and that* $k_1 = ... = k_m$. *Then, for all* $\mathbf{u}_0 = (u_{0,1}, ..., u_{0,m})$, *such that* $\mathbf{u}_0 \in (L^1(\Omega)^+ \cap (\mathbf{H}^1)^*)^m$, *the previous system has a non-negative global weak solution which satisfies for all* $T > 0$ *and* $i = 1, ..., m$,

$$u_i \in L^2(Q_T) \quad \text{and} \quad (1 + |u_i|)^\alpha \in L^2\big(0, T; H^1(\Omega)\big), \qquad \forall \alpha \in \left[0, \frac{1}{2}\right),$$

$$u_i \in L^\beta\big(0, T; W^{1,\beta}(\Omega)\big) \quad \text{and} \quad u_i \in L^\beta\big(0, T; L^\beta(\Gamma)\big), \qquad \forall \beta \in \left[1, \frac{d}{d-1}\right).$$

- C. G., Perthame B. Existence of a global weak solution for a reaction-diffusion problem with membrane conditions, *J. Evol. Equ.* (2020).

Effect of the membrane on patterns

Figure: Example of animals with spots and stripes.

Effect of the membrane on patterns

Figure: Example of animals with spots and stripes.

- **2 species**: u and v with different diffusion $(\boldsymbol{D_u}, \boldsymbol{D_v})$ and permeability coefficients $(\boldsymbol{k_u}, \boldsymbol{k_v})$

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- **2 species**: u and v with different diffusion $(\boldsymbol{D_u}, \boldsymbol{D_v})$ and permeability coefficients $(\boldsymbol{k_u}, \boldsymbol{k_v})$
- **reaction-diffusion membrane** problem with **Neumann** homogeneous boundary conditions

Turing instability

Theorem (Turing instability)

Consider the linearised reaction-diffusion system around the steady state $(\overline{u}, \overline{v})$ with $D_v > 0$ fixed. We assume $(\overline{u}, \overline{v})$ to be stable for the dynamical system and

$$\nu_D := \frac{D_{ur}}{D_{ul}} = \frac{D_{vr}}{D_{vl}}, \qquad \nu_K := \frac{k_u}{D_{ul}} = \frac{k_v}{D_{vl}} \qquad \text{and} \qquad \theta := \frac{D_{ul}}{D_{vl}} = \frac{D_{ur}}{D_{vr}}.$$

Moreover, let u be the activator and v the inhibitor. Then, for θ sufficiently small (that means D_u), the steady state $(\overline{u}, \overline{v})$ is linearly unstable.

- C. G. Effect of a membrane on diffusion-driven Turing instability, *preprint* (2021)

Numerical results

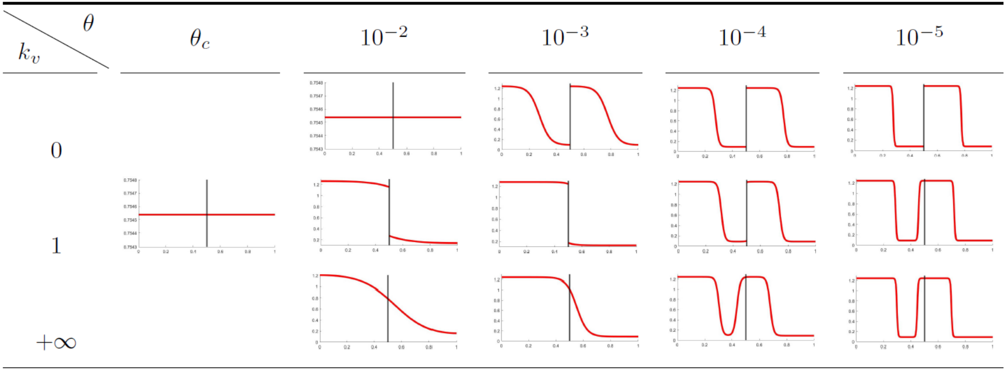

Figure: Summary of the evolution of patterns for the activator u varying θ and k_v (then, k_u).

Conclusions

- C. G., Perthame B. *J. Evol. Equ.* (2020).
  - Reaction-diffusion membrane problem with Dirichlet homogeneous conditions.
  - Existence and regularity theorem.
- C. G. *preprint* (2021)
  - Reaction-diffusion membrane problem with Neumann homogeneous conditions.
  - Turing instability: same conditions as for the case without membrane plus

$$\frac{D_{ur}}{D_{ul}} = \frac{D_{vr}}{D_{vl}}, \qquad \frac{k_u}{D_{ul}} = \frac{k_v}{D_{vl}}$$

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Thank you for the attention!

Main result

Definition

We define $\mathbf{H^1} = H^1_{0,\Gamma}(\Omega^1) \times H^1_{0,\Gamma}(\Omega^2)$ *as the Hilbert space of functions* $H^1(\Omega^1) \times H^1(\Omega^2)$ *satisfying Dirichlet homogeneous conditions on* Γ^λ, $\lambda = 1, 2$. *We endow it with the norm*

$$\|w\|_{\mathbf{H^1}} = \left(\|w^1\|^2_{H^1(\Omega^1)} + \|w^2\|^2_{H^1(\Omega^2)}\right)^{\frac{1}{2}}.$$

We let $(\cdot, \cdot)$ *be the inner product in* $\mathbf{H^1}$ *and* $\langle \cdot, \cdot \rangle$ *denote the pairing of* $\mathbf{H^1}$ *with its dual space.*

Theorem (Existence and regularity)

Assume (1)–(4) *and that* $k_1 = ... = k_m$. *Then, for all* $\boldsymbol{u}_0 = (u_{0,1}, ..., u_{0,m})$, *such that* $\boldsymbol{u}_0 \in (L^1(\Omega)^+ \cap (\mathbf{H^1})^*)^m$, *the previous system has a non-negative global weak solution in the sense of the above Definition which satisfies for all* $T > 0$ *and* $i = 1, ..., m$,

$$u_i \in L^2(Q_T) \quad \text{and} \quad (1 + |u_i|)^\alpha \in L^2\big(0, T; H^1(\Omega)\big), \qquad \forall \alpha \in \left[0, \frac{1}{2}\right),$$

$$u_i \in L^\beta\big(0, T; W^{1,\beta}(\Omega)\big) \quad \text{and} \quad u_i \in L^\beta\big(0, T; L^\beta(\Gamma)\big), \qquad \forall \beta \in \left[1, \frac{d}{d-1}\right).$$

Main result

Definition

We define a weak solution of the previous system as a function $\boldsymbol{u} = (u_1, ..., u_m)$ *such that for all* $T > 0$ *and* $i = 1, ..., m$, $u_i \in L^1(Q_T)$, $f_i(\boldsymbol{u}) \in L^1(Q_T)$ *and for* $\psi \in \mathcal{D}_i$, *it holds*

$$-\int_\Omega \psi(0,x)u_{0,i} + \int_{Q_T} u_i(-\partial_t\psi - D_i\Delta\psi) = \int_{Q_T} \psi f_i.$$

Theorem (Existence and regularity)

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Steps of the proof

1. *Regularisation process.*

   $\rightarrow \mathbf{u}^n = (u_1^n, ..., u_m^n)$, regularized solution of the approximate system with $\mathbf{f}^n$ satisfying the previous hypothesis.

2. *An L^2 a priori lemma.*

   $\rightarrow$ We extend the Laamri-Perthame a priori L^2 estimate of the solution given an L^1 initial data to the case of membrane conditions.

3. *Existence of a global weak supersolution.*

   $\rightarrow$Compactness result applied to $\mathbf{u}_n$. For all $i = 1, ..., m$,

$$-\int_{\Omega} \psi(0,x) u_{0,i}^n + \int_{Q_T} (-\psi_t u_i^n + D_i \nabla\psi \nabla u_i^n) + \int_0^T \int_{\Gamma} D_i k_i [u_i^n][\psi] = \int_{Q_T} \psi f_i^n.$$

   Goal: $n \rightarrow +\infty$
   $\rightarrow f_i^n$ does not converge in L^1
   $\rightarrow$ Truncation method: reaction terms are under control as $n \rightarrow +\infty$ and then truncation level to $+\infty$

4. *Existence of a global weak solution.*

Turing conditions

Lemma (Conditions for $w_n = z_n$)

Let

$$\nu_D := \frac{D_{ur}}{D_{ul}} = \frac{D_{vr}}{D_{vl}}, \qquad \nu_K := \frac{k_u}{D_{ul}} = \frac{k_v}{D_{vl}} \qquad \text{and} \qquad \theta := \frac{D_{ul}}{D_{vl}} = \frac{D_{ur}}{D_{vr}}.$$

A sufficient and necessary condition to have $w_n = z_n$ is the following relation

$$\frac{\lambda_n}{D_u} = \frac{\eta_n}{D_v}, \qquad \text{i.e.} \quad \frac{\eta_n}{D_{vl}} = \frac{\lambda_n}{D_{ul}} \quad \text{and} \quad \frac{\eta_n}{D_{vr}} = \frac{\lambda_n}{D_{ur}},$$

Proof

The eigenvalue problems of the Laplace operator with Neumann and membrane conditions for u and v are equivalent to a problem of the form

$$\begin{cases} -\frac{D}{D_l}\Delta\phi_n = \frac{\Lambda_n}{D_l}\phi_n, & \text{in } \Omega_l \cup \Omega_r \\ \partial_n\phi_n = 0, & \text{in } \Gamma_l \cup \Gamma_r \\ \partial_n\phi_{ln} = \nu_D \partial_n\phi_{rn} = \nu_K(\phi_{rn} - \phi_{ln}), & \text{in } \Gamma. \end{cases}$$