Analysis of two "Rolling carpet" strategies to eradicate an invasive species

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Joint work with Luis Almeida and Nicolas Vauchelet

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Introduction

Biological motivation

Mosquitoes are vectors of many diseases (Zicka, Denguee, chikungunya).
They are responsible of 700 000 deaths annually[1].
A strategy to avoid these epidemics is to eradicate such vectors.

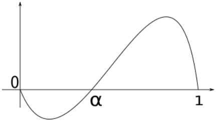
The task of this presentation is to present two strategies in order to eradicate an invasive species.

1. World Health Organization

Bistable Dynamics and Traveling Waves solutions

Assumption : Bistable dynamics

$$\underbrace{\partial_t u - \Delta u}_{\text{Diffusion}} = \underbrace{g(u)}_{\text{Growth}}$$



Natural solutions : the traveling waves solutions $\quad u(x,t) = \phi(x - c_0 t)$.

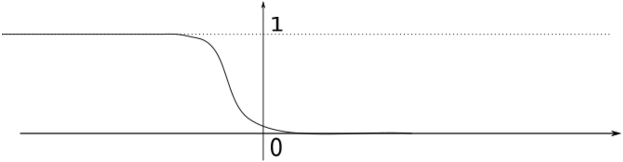
$$\partial_t u - \Delta u = g(u) \quad \longrightarrow \quad -c_0\phi' - \phi'' = g(\phi).$$

ϕ connects the two stable steady states :

$$\phi(-\infty) = 1 \quad \text{and} \quad \phi(+\infty) = 0.$$

The sense of propagation depends on $\mathrm{sign}(c_0) = \mathrm{sign}\left(\int_0^1 g(v)dv\right)$

Assumption : $\mathrm{sign}(c_0) > 0$: naturally, the mosquitoes invades the territory



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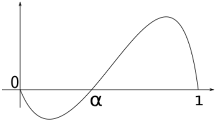

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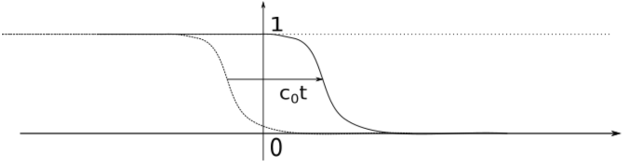

The rolling carpet strategy

The idea is to *act* on a finite interval $(0, L)$ and move this *action* like a rolling carpet in the opposite sens than the natural invasion traveling wave.

Aim of the work : generate a traveling wave with a negative speed solution of

$$\begin{cases} \partial_t u - \Delta u = g(u) 1_{\{x<-ct, x>L-ct\}} + Act(u) 1_{\{-ct<x<L-ct\}}, \\ u(-\infty) = 1, \\ u(+\infty) = 0 \end{cases}$$

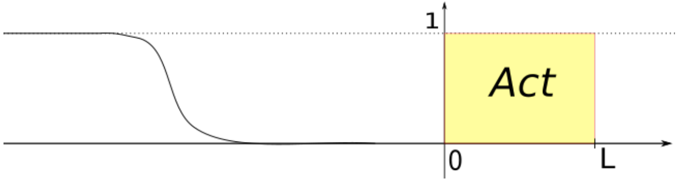

Fig. 1. The system at time $t = 0$

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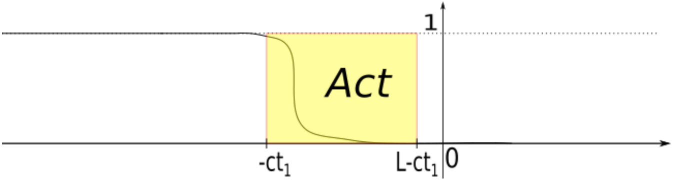

Fig. 1. The system at $t = t_1$

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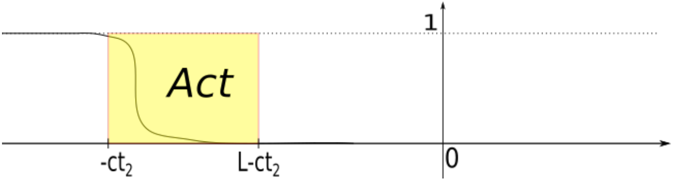


Fig. 1. The system at $t = t_2$

The rolling carpet strategy

The idea is to *act* on a finite interval $(0, L)$ and move this *action* like a rolling carpet in the opposite sens than the natural invasion traveling wave.

Aim of the work : generate a traveling wave with a negative speed solution of

$$\begin{cases} -c\phi_L' - \phi_L'' = g(\phi_L)1_{\{x<0,x>L\}} + Act(\phi_L)1_{\{0<x<L\}}, \\ \phi_L(-\infty) = 1 \qquad \phi_L(+\infty) = 0 \end{cases}$$

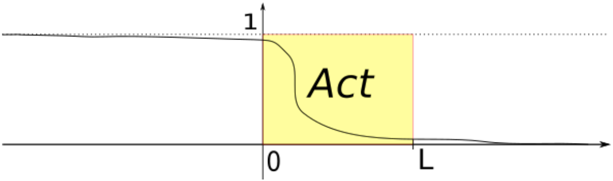


Fig. 1. The traveling wave solution ϕ_L

The killing strategy

The strategy and the hypothesis

The dynamics

$$\partial_t u - \Delta u = g(u)1_{\{x<-ct, x>L-ct\}} + Act(u)1_{\{-ct<x<L-ct\}}$$

Action : To kill many individuals and eggs in the interval $(-ct, L-ct)$

$$\partial_t u - \Delta u = \begin{cases} g(u) & \text{for } x < -ct,\ x > L+ct, \\ -\mu u & \text{for } x \in (-ct, L-ct). \end{cases}$$

Biological application : The use of insecticide

Hypothesis : The death rate in $(-ct, L-ct)$ is higher than everywhere else

$$i.e. \quad -\mu u < g(u).$$

Free parameter : The size L

Recalling the equation :

$$\begin{cases} -c\phi_L' - \phi_L'' = g(\phi_L)1_{\{x<0,x>L\}} - \mu\phi_L 1_{\{0<x<L\}}, \\ \phi_L(-\infty) = 1, \qquad \phi_L(+\infty) = 0 \end{cases}$$

Theorem (Almeida-L.-Vauchelet)

For every speed $c \leq 0$, there exists a critical size $\Lambda(c)$ such that

1. If $L < \Lambda(c)$ then the system does **not** admit a traveling wave solution ϕ_L
2. If $L > \Lambda(c)$ then the system admits a decreasing traveling wave ϕ_L.

Assuming that $g(u) = u(1-u)(u-\alpha)$, then the system admits a solution for $L = \Lambda(c)$ if and only if

$$-2\sqrt{g'(\alpha)} < c \leq 0.$$

Moreover, we have $\lim\limits_{\substack{L\to\Lambda(c) \\ L>\Lambda(c)}} u'(L) = 0.$

✎ [Berestycki, Rodriguez Ryzhik, 2013] Special case $c = 0$ (no moving interval)

Main tool :

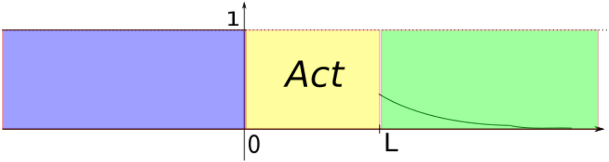
The equation is "autonomous by part"

$\downarrow$

Allow to use the usual tools of autonomous equation

Application 1 : Construction of a super-solution

$$\overline{\phi}(x) = \begin{cases} 1 & \text{for } x < 0, \\ \\ \phi_*(x) & \text{for } x > L \end{cases}$$



Ingredient of the proof : Sub and Super-soutions

Main tool :

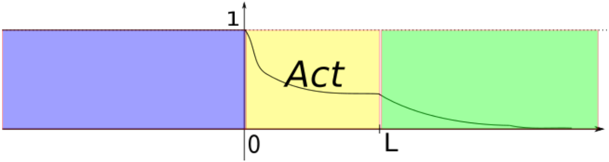
The equation is "autonomous by part"

$\downarrow$

Allow to use the usual tools of autonomous equation

Application 1 : Construction of a super-solution

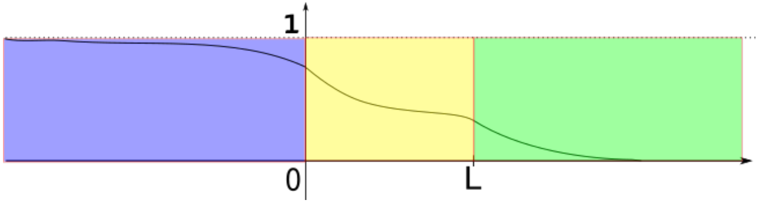
$$\overline{\phi}(x) = \begin{cases} 1 & \text{for } x < 0, \\ \dfrac{c(r_+ e^{r_-(x-L)} - r_- e^{r_+(x-L)})}{r_+ - r_-} & \text{for } 0 < x < L, \\ \phi_*(x) & \text{for } x > L \end{cases}$$



Ingredient of the proof : characterization of the critical size

Application 2 : Construct a new solution with a minimal size L

Step 1 Autonomous equation for $x > L \Rightarrow \begin{cases} \text{Uniqueness of the tails of } \phi_L, \\ \phi_L \text{ is decreasing.} \end{cases}$



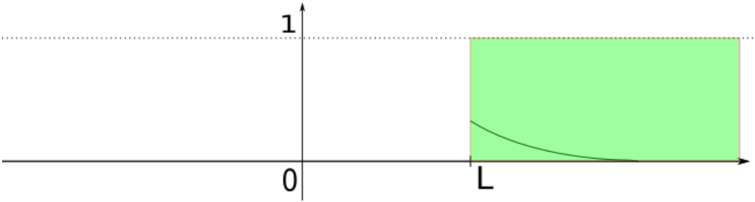
Ingredient of the proof : characterization of the critical size

Application 2 : Construct a new solution with a minimal size L

Step 1 ϕ_L decreasing + uniqueness of the tails

Step 2 If $\phi_L'(L) < 0$, then consider the *unique tail*

$$\begin{cases} -cv' - v'' = g(v), \\ v(L) = \phi_L(L), \ v'(L) = \phi_L'(L). \end{cases}$$

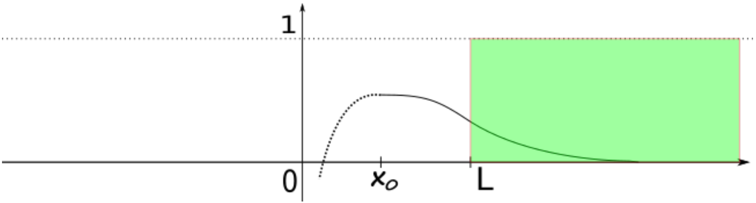


Application 2 : Construct a new solution with a minimal size L

Step 1 ϕ_L decreasing + uniqueness of the tails

Step 2 Consider the unique tail v

Step 3 We let $v(x)$ evolves for $x < L$ until $v'(x_0) = 0$, $v''(x_0) < 0$



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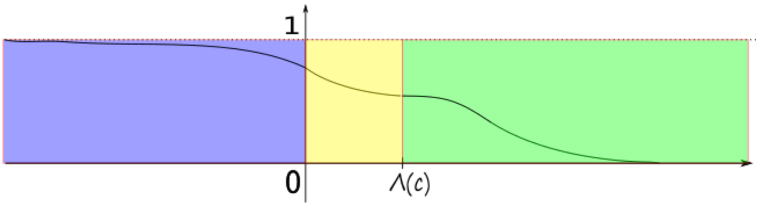
Step 4 We connect $v_{|]x_0,+\infty[}$ with the solution of the system

$\Rightarrow$ Lead to the solution with the minimal size $\Lambda(c)$



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Step 3 We let $v(x)$ evolves for $x < L$ until $v'(x_0) = 0$ **Existence of x_0 ?**

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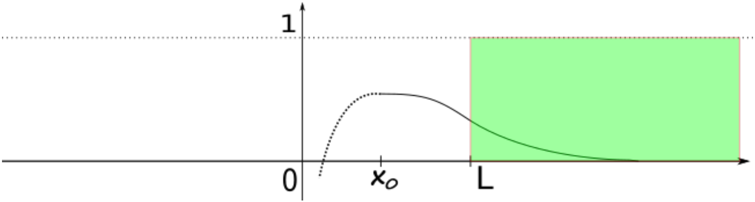


Answer : Yes $\Leftrightarrow |c| < 2\sqrt{g'(\alpha)}$
(Rely on the existence of Fisher-KPP traveling wave connexting α to 0)

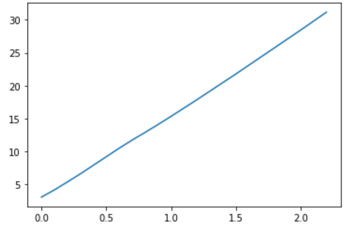
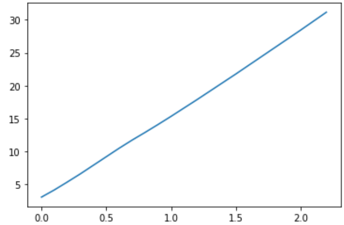


Fig. 2. Numerical computations of the functions $\Lambda(c)$



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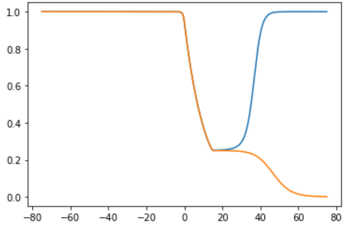

Fig. 3. Numerical solution for $c = 1$, $L = \Lambda(1)$ (orange) and $L = \Lambda(1) - 10^{-4}$ (blue)

The sterile males strategy

The strategy and the hypothesis

Action : To release sterile males m_S in $(-ct, L-ct)$

Remark : The dynamics must include the dynamics of the sterile males

Hypothesis : $\#\{$ Fertile Females $\} \sim \#\{$ Fertile Males $\}$

New dynamics - Fertile Females f + Sterile males m_S

$$\begin{cases} \partial_t f - \partial_{xx} f = g(f, m_S), \\ \partial_t m_S - \partial_{xx} m_S = \mathrm{M}1_{\{-ct<x<L-ct\}} - \mu_M m_S \end{cases}$$

with • $g(f,0)$ bi-stable,

- $(m_s \mapsto g(f, m_s))$ decreasing,
- $g(f, m_s) \underset{m\to+\infty}{\longrightarrow} -\mu_F f$,
- $g(0, m_s) = 0$.

Free parameter : The size L & the released quantity M

The main result

Aim : Obtain a traveling wave solution $\phi_{L,M}$

$$\begin{cases} -c\phi'_{L,M} - \phi''_{L,M} = g(\phi_{L,M}, m_S), \\ -cm'_S - m''_S = \mathrm{M}1_{\{0<x<L\}} - \mu_M m_S \\ \phi_{L,M}(-\infty) = 1, \quad \phi_{L,M}(\infty) = 0. \end{cases}$$

Theorem (Almeida-L.-Vauchelet)

For every speed $c \leq 0$ and size $L > 0$, there exists a critical number of mosquitoes $\Pi(c, L)$ such that

1. If $M < \Pi(c, L)$ then the system does **not** admit a solution $\phi_{L,M}$.
2. If $M > \Pi(c, L)$ then the system admits a traveling wave $\phi_{L,M}$.

Moreover, we have

$$\lim_{L\to 0} \Pi(c, L) = +\infty, \quad \liminf_{L\to+\infty} \Pi(c, L) > 0 \quad \text{and} \quad \lim_{c\to-\infty} \Pi(c, L) = +\infty.$$

✎ [Almeida, Estrada, Vauchelet, 2021] Special case $c = 0$ (no moving interval) for (E, F, M)

New difficulty : The system is fully non-autonomous

Full Characterization of m_S : $m_S(x) = ML \int_{\mathbb{R}} \frac{\mathrm{sinc}\left(\frac{L\xi}{2}\right) e^{2i\pi\xi x + i\pi L}}{2[4\pi^2\xi^2 + 2i\pi c\xi + \mu_S]} d\xi.$

Corollary

$m_S \underset{M \to +\infty}{\longrightarrow} +\infty$ uniformly locally.

Consequence : If $M >> 1$ then

$$g(f,m) \leq \begin{cases} g(f,0) & \text{for } x < 0,\ x > L_*, \\ -\mu f & \text{for } 0 < x < L_*. \end{cases}$$

$\rightarrow$ Allow to use the *killing strategy Theorem* to obtain a super-solution

Ingredient of the proof

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Remark 1 : The sub-solution is more tricky to obtain
Remark 2 : We did not succeed in characterizing the critical number Π

Numerical result

We perform numerical simulations of

$$\partial_t f - \partial_{xx} f = g(f, m_S),$$

for $c = -0.05$, $M = 20000$ and two sizes of L.

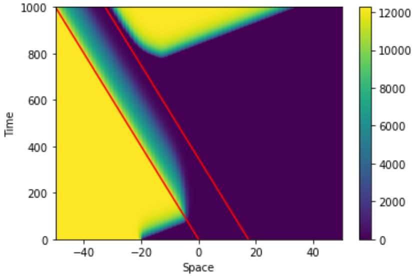


Fig. 4. $L = 17.4$

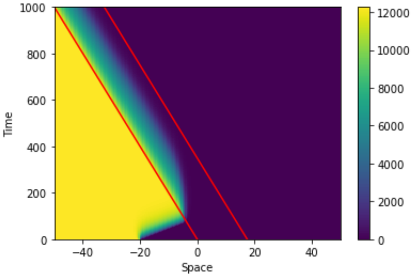
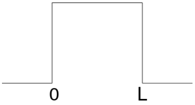
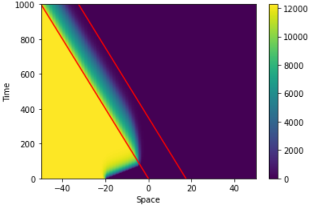
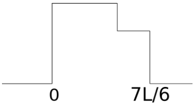
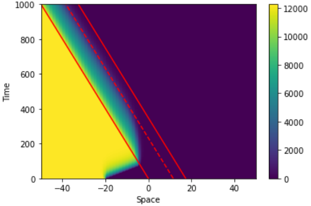


Fig. 5. $L = 17.49$

Perspectives

Perspectives

1. To obtain a characterization of $\Pi(c, L)$
2. To minimize the number of needed mosquitoes : $\mathcal{N}(c) = L \times \Pi(c, L)$
3. To consider other types of released (less mosquitoes are needed near L than 0) :

Released	Number $\mathcal{N}$	Solution
	$\Pi(c, L) \times L$	
	$\dfrac{11\Pi(c, L) \times L}{12}$	

Thank you for your attention