How to use asymptotics to derive a low-dimensional and mathematically well posed model of falling film flows.

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Thin Films Between Physical Basis and Phenomena

Thin liquid film

Modeling

Solution

Physical Forces

Inertia

viscosity

surface tension

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Scaling For Thin Liquid Films

\[ \begin{align*}
\nabla \cdot \mathbf{u} &= 0, \\
\rho (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) &= -\nabla p + \rho \mathbf{g} \mathbf{e} + \mu \Delta \mathbf{u} \quad \text{Inertia} \quad \text{Pressure} \quad \text{Viscosity}
\end{align*} \]

\[ \text{In Long wave limit: } \epsilon \text{ very small } \implies \text{Drag-gravity Regime} \]

\[ \text{What if } \epsilon \text{ is not too small } \implies \text{What physical effect to count on, and what to ignore? } \implies \text{Drag-Inertia Regime} \]

\[ \epsilon Fr^2 \text{Inertia} = \epsilon (\text{Pressure} + \text{Surface Tension}) + \frac{Fr^2}{Re} (\text{Viscosity} + \text{Gravity}) \]

\[ \text{Ansatz: } Re = O(1) \]

\[ \{ \text{Reynold nb : Re, Froude nb : Fr, smallness parameter : } \epsilon \} \]

+ \text{B.C: Surface Tension}
Weighted Residual Method [1]

- Start by choosing the velocity profile
  1. At first order using Gauge condition

    \[ u = 3Ug_0 \text{ where } \int_0^h u = hU \]

- Exact profile by adding the correction

  \[ u = 3Ug_0 + \epsilon \tilde{u} \iff \int_0^h \tilde{u} = 0 \]

How to figure out \( \tilde{u} \) and other variables??

- Use Weighted residual method
  1. From Mom EQ \( \partial_{zz} u = 3U \partial_{zz} g_0 + \epsilon \partial_{zz} \tilde{u} \)
  2. Choose \( G \) such that by integration by parts

    \[
    \int_0^h \partial_{zz} \tilde{u} \times G = B.C + \int_0^h \tilde{u} \partial_{zz} G = C \int_0^h \tilde{u} = 0
    \]

- \( \tilde{u} \) eliminates from big orders of Mom EQ \( \rightarrow \) Solve obtained system for \( h \) and \( U \)

"C. Ruyer-Quil, P. Manneville,
Improved modeling of flows down inclined planes, 2000"
Pros of WRM
- Very good numerical results compared to Navier Stokes system results
- Better representation of Drag-inertia regimes where $\epsilon$ not very small

Cons
- No energy equation
- Lose hyperbolicity

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S. Chakraborty, P.-K. Nguyen, C. Ruyer-Quil and V. Bontozoglou,
Extreme solitary waves on falling liquid films, 2014
Propositions inspired by [1]

\[ \epsilon \text{Inertia terms} = \text{Gravity} + \text{Viscosity} + \epsilon^2 \text{Dispersive terms} \]

- Viscosity terms give good eigenvalues \( \rightarrow \) Preserve them
- Change Inertia terms using Long wave expansion so that we
  1. Preserve the good representation Drag-inertia regime by WRM
  2. Obtain dissipative energy
  3. Obtain Linear stability

"[1] G. L. Richard, C. Ruyer-Quil and J. P. Vila,
A three-equation model for thin films down an inclined plane, 2016
First Approach

- **Comparison:**
  Inertia terms from depth-integrating, i.e Shallow Water Model:

  \[
  \partial_t (hU) + \partial_x \left( \int_0^h u^2 \right) \quad \text{where} \quad \int_0^h u^2 = \int_0^h (U + u - U)^2 = hU^2 + O(\epsilon)
  \]

- **Idea:**
  Better approximation of \( \int_0^h (u - U)^2 \) \( \Rightarrow \) Claim there exists \( \psi_1 \)

  \[
  \int_0^h (u - U)^2 = \frac{h^3}{5} (U - 6 \frac{s_1}{h})^2 + O(\epsilon^2) \quad \ldots \text{4 Eq Model by WRM}
  \]

  \[
  = \frac{h^3}{5} \psi_1^2 + O(\epsilon^2)
  \]

- **Velocity profile**

  \[
  \psi_1 = U - 6 \frac{s_1}{h} \quad \Rightarrow \quad u = F(U, \psi_1)
  \]

- **Procedure:**
  Derive 3 equations on \( h, U, \Psi_1 \) using the residues

  \[
  R_1 = \langle \tilde{u}, 1 \rangle = \int_0^h \tilde{u} \, dz = 0 \]

  \[
  R_2 = \langle \tilde{u}, g_0 \rangle = \int_0^h \tilde{u}g_0 \, dz = 0.
  \]
Transforming Inertia Terms: preserving consistency

- By WRM, we solve above equations for $\partial_t U$ and $\partial_t \psi_1$

\[
\begin{align*}
I_U & \sim \frac{1}{\epsilon \text{Re}} \left( \frac{14}{15} (\lambda h - 3U/h) + \frac{21}{5} (\psi_1 - U/h) \right) - \frac{14}{15} \left( \frac{\cos \theta}{F r^2} h \partial_x h - \frac{\kappa}{F r^2} h \partial_x^3 h \right) \\
I_{\psi_1} & \sim \frac{1}{\epsilon \text{Re}} \left( \frac{1}{3} (\lambda - 3U/h^2) + \frac{21}{h} (\frac{U}{h} - \psi_1) \right) - \frac{1}{3} \left( \frac{\cos \theta}{F r^2} \partial_x h - \frac{\kappa}{F r^2} \partial_x^3 h \right)
\end{align*}
\]

- Using definition of $\psi_1$ and asymptotic expansions we prove

\[
I_U \sim \partial_t (hU) + \partial_x (hU^2 + \frac{h^3 \psi_1^2}{5}) + O(\epsilon^2) \\
I_{\psi_1} \sim \partial_t (h\psi_1) + \partial_x (hU \psi_1) - \frac{1}{7} \frac{\partial_x (h^4 \psi_1^3)}{h^2 \psi_1} + O(\epsilon)
\]
Resulting System

\[ \partial_t h + \partial_x (hU) = 0, \]

\[ \partial_t (hU) + \partial_x (hU^2 + \frac{1}{5} h^3 \psi_1^2) = \frac{1}{\epsilon Re} \left( \frac{14}{15} \left( \lambda h - \frac{3U}{h} \right) + \frac{21}{5} \left( \psi_1 - \frac{U}{h} \right) \right) \]

\[ - \frac{14}{15} \left( \frac{\cos \theta}{Fr^2} h \partial_x h - \frac{\kappa}{Fr^2} h \partial^3_x h \right) + \frac{\epsilon}{Re} D_1, \]

\[ \partial_t (h \psi_1) + \partial_x (hU \psi_1) - \frac{1}{7} \frac{\partial_x (h^4 \psi_1^3)}{h^2 \psi_1} = \frac{1}{\epsilon Re} \left( \frac{1}{3} \left( \lambda - \frac{3U}{h^2} \right) + \frac{21}{h} \left( \frac{U}{h} - \psi_1 \right) \right) \]

\[ - \frac{1}{3} \left( \frac{\cos \theta}{Fr^2} \partial_x h - \frac{\kappa}{Fr^2} \partial^3_x h \right) + \frac{\epsilon}{Re} D_2, \]

(2)

- Term in violet approximation of \( \int_0^h (u - U)^3 \)
- **Pros:** Dissipative energy
Numerical results of the Solitary wave test using AUTO

Figure 1: the height and velocity speed showing certain stability for sufficiently high Reynold number
Figure 2: profiles of $h$ and $\psi_1$ for Re=20, Ka=3400
• One step further: Claim there exists $\psi_1$, $\psi_2$ and $\psi_3$

$$\int_0^h (u - U)^2 = \frac{h^3}{5} (U - 6 \frac{s_1 + s_2}{h})^2 + 4h^3 (\frac{s_1}{2} - \frac{3}{2} s_2)^2 + \frac{225}{13h} s_2^2 + O(\epsilon^2)$$

$$= \frac{h^3}{5} \psi_1^2 + 4h^3 \psi_2^2 + \frac{225}{13} h^3 \psi_3^2 + O(\epsilon^2)$$

• Velocity profile

$$u = F(U, \psi_1, \psi_2)$$

• Procedure: Derive 4 equations on $h$, $U$, $\Psi_{1,2}$, ($\Psi_3 = F(h, U, \Psi_1, \Psi_2)$) using

$$R_1 = <\tilde{u}, 1> = \int_0^h \tilde{u} \, dz = 0$$
$$R_2 = <\tilde{u}, g_0> = \int_0^h \tilde{u} g_0 \, dz = 0$$

$$R_3 = <\tilde{u}, g_1> = \int_0^h \tilde{u} g_1 \, dz = 0.$$
Goal: Reformulating inertia terms→Dissipative Energy+linear stability

- First Eq: Easy

\[ \partial_t (hU) + \partial_x (hU^2 + \frac{h^3 \psi_1^2}{5} + 4h^3 \psi_2^2) + O(\epsilon^2). \]

- Second Eq:

1. First approach as with \( \psi_1 \) with the term \(-\frac{1}{7} \partial_x (h^3 \psi_1^4)\):
   failed to obtain the secondary fixed point

2. Second approach:
   1. Preserve transport part for the sake of energy
   2. Express different gradients that get canceled in mechanical energy equation

\[ I_{\psi_1} = \partial_t (h \psi_1) + \partial_x (hU \psi_1) + \frac{20}{h^2 \psi_1} \left( \partial_x (A1 h^3 U \psi_1^2 + B1 h^4 \psi_1^3 + C1 h^4 \psi_1^2 \psi_2) \right. \]
\[ \left. - DDh^4 \psi_2 \psi_1 \partial_x \psi_1 - EEh^3 \psi_2 \psi_1^2 \partial_x h - FFh^3 \psi_2 \psi_1 \partial_x U - G Gh^2 \psi_2 \psi_1 U \partial_x h \right) \]
\[ + O(\epsilon) \]

\[ I_{\psi_2} = \partial_t (h \psi_2) + \partial_x (hU \psi_2) + \frac{1}{h^2 \psi_2} \left( \partial_x (A2 h^3 U \psi_2^2 + B2 h^4 \psi_1 \psi_2^2 + C2 h^4 \psi_3^3) \right. \]
\[ \left. - DDh^4 \psi_2 \psi_1 \partial_x \psi_1 - EEh^3 \psi_2 \psi_1^2 \partial_x h - FFh^3 \psi_2 \psi_1 \partial_x U - G Gh^2 \psi_2 \psi_1 U \partial_x h \right) \]
\[ + O(\epsilon) \]
End of presentation

Thank You!