How to use asymptotics to derive a low-dimensional and mathematically well posed model of falling film flows.

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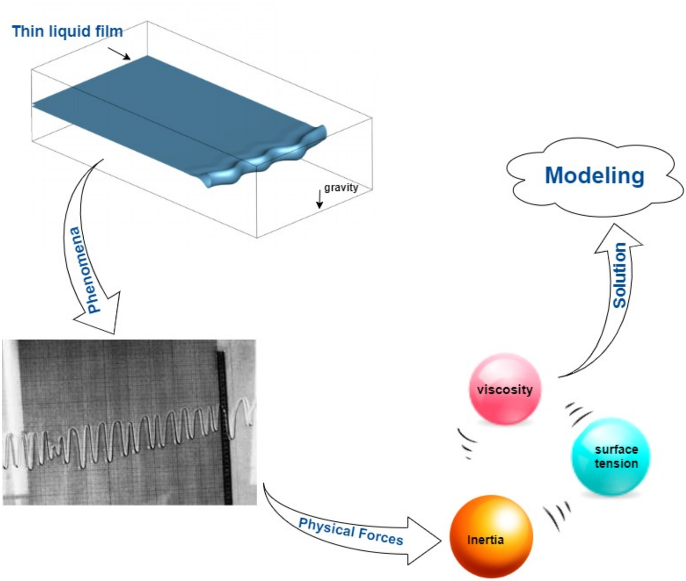
Thin Films Between Physical Basis and Phenomena



Scaling For Thin Liquid Films

$$\begin{cases} \nabla \cdot \mathbf{u} & = 0, \\ \underbrace{\rho(\,\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u})}_{\text{Inertia}} & = - \underbrace{\nabla p}_{\text{Pressure}} + \rho g\,\vec{e} + \underbrace{\mu\Delta\mathbf{u}}_{\text{Viscosity}} \end{cases} . \xrightarrow{Scaling} \begin{cases} \text{Reynold nb : Re,} \\ \text{Froude nb : Fr,} \\ \text{smallness parameter :}\epsilon \end{cases}$$

+**B.C**: Surface Tension

Ansatz: $Re = O(1)$

$$\epsilon Fr^2 \text{Inertia} = \epsilon\big(\text{Pressure} + \text{Surface Tension}\big) + \frac{Fr^2}{Re}(\text{Viscosity+Gravity})$$

- In Long wave limit: ϵ very small $\Longrightarrow$ Drag-gravity Regime
- What if ϵ is not too small $\longrightarrow$ What physical effect to count on, and what to ignore ?? $\Longrightarrow$ Drag-Inertia Regime

Weighted Residual Method [1]

- Start by choosing the velocity profile
  1. At first order using Gauge condition

$$u = 3U g_0 \text{ where } \underbrace{\int_0^h u = hU}_{\text{Gauge cnd}}$$

  2. Exact profile by adding the correction

$$u = 3U g_0 + \epsilon \tilde{u} \qquad \Longleftrightarrow \int_0^h \tilde{u} = 0$$

How to figure out $\tilde{u}$ and other variables??

- Use Weighted residual method
  1. From Mom EQ $\partial_{zz} u = 3U\, \partial_{zz} g_0 + \epsilon\, \partial_{zz} \tilde{u}$
  2. Choose G such that by integration by parts

$$\int_0^h \partial_{zz}\tilde{u} \times G = B.C + \int_0^h \tilde{u}\, \underbrace{\partial_{zz} G}_{=cst} = C \int_0^h \tilde{u} = 0$$

- $\tilde{u}$ eliminates from big orders of Mom EQ –> Solve obtained system for h and U

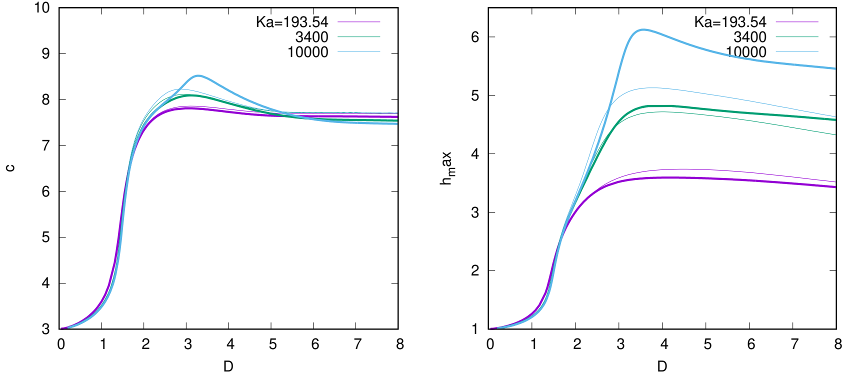
" C. Ruyer-Quil, P. Manneville,
Improved modeling of flows down inclined planes, 2000

Pros of WRM

- Very good numerical results compared to Navier Stokes system results
- Better representation of Drag-inertia regimes where ϵ not very small

Cons

- No energy equation
- Lose hyperbolicity



" S. Chakraborty, P.-K. Nguyen, C. Ruyer-Quil and V. Bontozoglou, Extreme solitary waves on falling liquid films, 2014

Propositions inspired by [1]

$$\epsilon \text{Inertia terms} = \text{Gravity} \; + \; \text{Viscosity} + \epsilon^2 \; \text{Dispersive terms}$$

- Viscosity terms give good eigenvalues –> Preserve them
- Change Inertia terms using Long wave expansion so that we
  1. Preserve the good represention Drag-inertia regime by WRM
  2. Obtain dissipative energy
  3. Obtain Linear stability

"[1] G. L. Richard, C. Ruyer-Quil and J. P. Vila,
A three-equation model for thin films down an inclined plane, 2016

First Approach

- **Comparison:**
  Inertia terms from depth-integrating, i.e Shallow Water Model:

$$\partial_t(hU) + \partial_x(\int_0^h u^2) \quad \text{where} \quad \int_0^h u^2 = \int_0^h (U + u - U)^2 = hU^2 + O(\epsilon)$$

- **Idea:**
  Better approximation of $\int_0^h (u-U)^2$ –> Claim there exists ψ_1

$$\int_0^h (u-U)^2 = \frac{h^3}{5}(U - 6\frac{s_1}{h})^2 + O(\epsilon^2) \qquad \textbf{...4 Eq Model by WRM}$$
$$= \frac{h^3}{5}\psi_1^2 + O(\epsilon^2)$$

- **Velocity profile**

$$\psi_1 = U - \frac{6s_1}{h} \implies u = F(U, \psi_1)$$

- **Procedure:**
  Derive 3 equations on $h,\ U,\ \Psi_1$ using the residues

$$R_1 =< \tilde{u}, 1 >= \int_0^h \tilde{u}\ dz = 0 \qquad R_2 =< \tilde{u}, g_0 >= \int_0^h \tilde{u} g_0\ dz = 0. \qquad (1)$$

Transforming Inertia Terms: preserving consistency

- By WRM, we Solve above equations for $\partial_t U$ and $\partial_t \psi_1$

$$\underbrace{h\,\partial_t U + ...}_{I_U} = \frac{1}{\epsilon Re}\left(\frac{14}{15}(\lambda h - \frac{3U}{h}) + \frac{21}{5}(\psi_1 - \frac{U}{h})\right) - \frac{14}{15}\left(\frac{\cos\theta}{Fr^2} h\,\partial_x h - \frac{\kappa}{Fr^2} h\,\partial_x^3 h\right)$$

$$\underbrace{h\,\partial_t \psi_1 + ..}_{I_{\psi_1}} = \frac{1}{\epsilon Re}\left(\frac{1}{3}(\lambda - \frac{3U}{h^2}) + \frac{21}{h}(\frac{U}{h} - \psi_1)\right) - \frac{1}{3}\left(\frac{\cos\theta}{Fr^2}\,\partial_x h - \frac{\kappa}{Fr^2}\,\partial_x^3 h\right)$$

- Using definition of ψ_1 and asymptotic expansions we prove

$$I_U \sim \partial_t(hU) + \underbrace{\partial_x(hU^2 + \frac{h^3\psi_1^2}{5})}_{\partial_x(\int_0^h u^2)} + O(\epsilon^2)$$

$$I_{\psi_1} \sim \partial_t(h\psi_1) + \partial_x(hU\psi_1) - \frac{1}{7}\frac{\partial_x(h^4\psi_1^3)}{h^2\psi_1} + O(\epsilon)$$

Resulting System

$$
\begin{aligned}
&\partial_t h + \partial_x(hU) = 0,\\
&\partial_t(hU) + \partial_x(hU^2 + \frac{1}{5}h^3\psi_1^2) = \frac{1}{\epsilon Re}\left(\frac{14}{15}(\lambda h - \frac{3U}{h}) + \frac{21}{5}(\psi_1 - \frac{U}{h})\right)\\
&\qquad - \frac{14}{15}\left(\frac{\cos\theta}{Fr^2}h\,\partial_x h - \frac{\kappa}{Fr^2}h\,\partial_x^3 h\right) + \frac{\epsilon}{Re}\mathcal{D}_1,\\
&\partial_t(h\psi_1) + \partial_x(hU\psi_1) - \frac{1}{7}\,\frac{\partial_x(h^4\psi_1^3)}{h^2\psi_1} = \frac{1}{\epsilon Re}\left(\frac{1}{3}(\lambda - \frac{3U}{h^2}) + \frac{21}{h}(\frac{U}{h} - \psi_1)\right)\\
&\qquad - \frac{1}{3}\left(\frac{\cos\theta}{Fr^2}\,\partial_x h - \frac{\kappa}{Fr^2}\,\partial_x^3 h\right) + \frac{\epsilon}{Re}\mathcal{D}_2,
\end{aligned}
\tag{2}
$$

- Term in violet approximation of $\int_0^h (u-U)^3$
- **Pros:** Dissipative energy

Numerical results of the Solitary wave test using AUTO

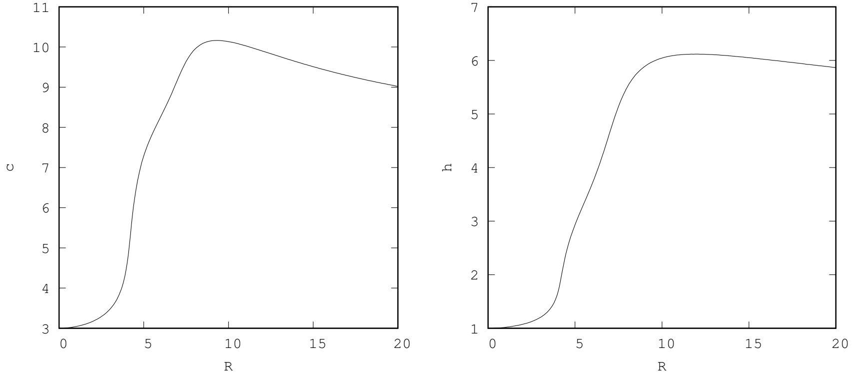


Figure 1: the height and velocity speed showing certain stability for sufficiently high Reynold number

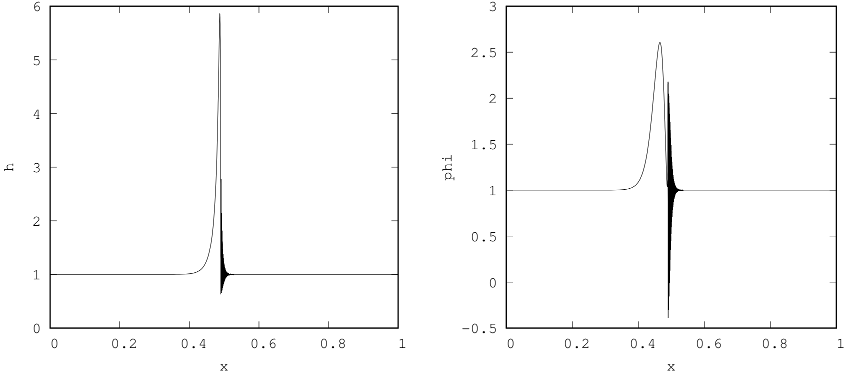


Figure 2: profiles of h and ψ_1 for Re=20, Ka=3400

- **One step further:** Claim there exists ψ_1, ψ_2 and ψ_3

$$\int_0^h (u-U)^2 = \frac{h^3}{5}(U - 6\frac{s_1+s_2}{h})^2 + 4h^3(s_1 - \frac{3}{2}s_2)^2 + \frac{225}{13h}s_2^2 + O(\epsilon^2)$$
$$= \frac{h^3}{5}\psi_1^2 + 4h^3\psi_2^2 + \frac{225}{13}h^3\psi_3^2 + O(\epsilon^2)$$

- **Velocity profile**

$$u = F(U, \psi_1, \psi_2)$$

- **Procedure:** Derive 4 equations on h, U, $\Psi_{1,2}$, $(\Psi_3 = F(h, U, \Psi_1, \Psi_2))$ using

$$R_1 =< \tilde{u}, 1 >= \int_0^h \tilde{u}\; dz = 0 \qquad R_2 =< \tilde{u}, g_0 >= \int_0^h \tilde{u} g_0\; dz = 0$$
$$R_3 =< \tilde{u}, g_1 >= \int_0^h \tilde{u} g_1\; dz = 0. \tag{3}$$

Goal: Reformulating inertia terms–>Dissipative Energy+linear stability

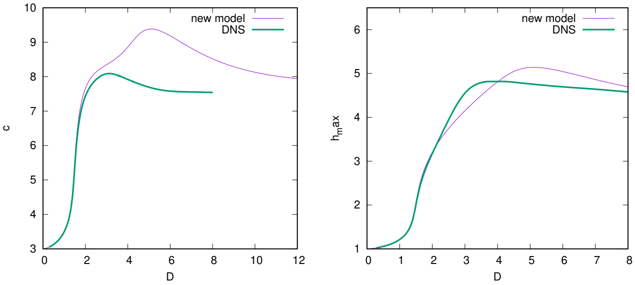
- First Eq: Easy

$$\partial_t(hU) + \partial_x(hU^2 + \frac{h^3\psi_1^2}{5} + 4h^3\psi_2^2) + O(\epsilon^2).$$

- Second Eq:
  1. fist approach as with ψ_1 with the term $-\frac{1}{7}\,\partial_x(h^3\psi_1^4)$:
     failed to obtain the secondary fixed point
  2. Second approach:
     1. Preserve transport part for the sake of energy
     2. Express different gradients that get canceled in mechaical energy equation

$$
\begin{aligned}
I_{\psi_1} &= \partial_t(h\psi_1) + \partial_x(hU\psi_1) + \frac{20}{h^2\psi_1}\Big(\partial_x(A1h^3U\psi_1^2 + B1h^4\psi_1^3 + C1h^4\psi_1^2\psi_2)\\
&\quad -DDh^4\psi_2\psi_1\,\partial_x\psi_1 - EEh^3\psi_2\psi_1^2\,\partial_x h - FFh^3\psi_2\psi_1\,\partial_x U - GGh^2\psi_2\psi_1 U\,\partial_x h\Big)\\
&\quad + O(\epsilon)\\
I_{\psi_2} &= \partial_t(h\psi_2) + \partial_x(hU\psi_2) + \frac{1}{h^2\psi_2}\Big(\partial_x(A2h^3U\psi_2^2 + B2h^4\psi_1\psi_2^2 + C2h^4\psi_2^3)\\
&\quad -DDh^4\psi_2\psi_1\,\partial_x\psi_1 - EEh^3\psi_2\psi_1^2\,\partial_x h - FFh^3\psi_2\psi_1\,\partial_x U - GGh^2\psi_2\psi_1 U\,\partial_x h\Big)\\
&\quad + O(\epsilon)
\end{aligned}
\tag{4}
$$

Numerical Result




End of presentation
Thank You!