Weak and Strong Solution for a Magnetohydrodynamic Problem

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Introduction

Magnetohydrodynamics (MHD) is the academic discipline which studies the dynamics of electrically conducting fluids. Examples of such fluids include plasmas, liquid metals, and salt water. The word magnetohydrodynamics (MHD) is derived from magneto meaning magnetic field, and hydro meaning liquid, and dynamics meaning movement.
The field of MHD was initiated by Hannes Alfvén, for which he received the Nobel Prize in Physics in 1970.

Introduction

The set of equations which describe MHD are a combination of the Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. The equations are non-linearly coupled via Ohm's law and the Lorentz force.

The mathematical problem

$$
(\mathcal{S})\begin{cases}
-\nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} - \dfrac{1}{\rho\mu}(\boldsymbol{B}\cdot\nabla)\,\boldsymbol{B} + \dfrac{1}{2\rho\mu}\nabla\left(|\boldsymbol{B}|^2\right) + \dfrac{1}{\rho}\nabla\pi = \boldsymbol{f} & \text{in } \Omega,\\
-\lambda\Delta \boldsymbol{B} + (\boldsymbol{u}\cdot\nabla)\,\boldsymbol{B} - (\boldsymbol{B}\cdot\nabla)\,\boldsymbol{u} = \boldsymbol{k} & \text{in } \Omega,\\
\operatorname{div}\ \boldsymbol{u} = 0, \quad \operatorname{div}\ \boldsymbol{B} = 0 & \text{in } \Omega,\\
\boldsymbol{u} = \boldsymbol{0}, \quad \boldsymbol{B}\cdot\boldsymbol{n} = 0, \quad \textbf{curl}\,\boldsymbol{B}\times\boldsymbol{n} \ = \boldsymbol{0} & \text{on } \Gamma,\\
\displaystyle\int_{\Sigma_j} \boldsymbol{B}\cdot\boldsymbol{n} = 0,\ 1\leqslant j\leqslant J. &
\end{cases}
$$

MHD problem

We suppose that Ω is a bounded open set of $\mathbb{R}^3$ of class $\mathcal{C}^{1,1}$ and possibly non simply connected with a boundary Γ.
We denote

$$\Gamma = \bigcup_{i=0}^{I} \Gamma_i$$

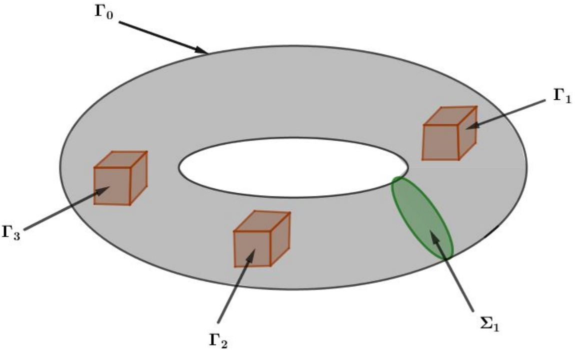
where Γ_i are the connected components of Γ.
We suppose that there exists J connected open surfaces Σ_j, called 'cuts', contained in Ω, the boundary of each Σ_j is contained in Γ and

$$\overline{\Sigma_i} \cap \overline{\Sigma_j} = \emptyset, \qquad i \neq j.$$

$\Omega^\circ = \Omega \setminus \bigcup_{j=1}^{J} \Sigma_j$ is simply-connected.

Domain

For $I = 3$ and $J = 1$.



Domain Ω

Properties of the functional spaces

We define the following Banach spaces, for $1 < p < \infty$:

$$\boldsymbol{H}^p(\mathbf{curl}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\},$$

$$\boldsymbol{H}^p(\mathrm{div}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\, \boldsymbol{v} \in L^p(\Omega)\},$$

$$\boldsymbol{X}^p(\Omega) = \boldsymbol{H}^p(\mathbf{curl}, \Omega) \cap \boldsymbol{H}^p(\mathrm{div}, \Omega)$$

Properties of the functional spaces

and their subspaces:

$$\boldsymbol{H}_0^p(\mathbf{curl},\,\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{H}^p(\mathbf{curl},\,\Omega);\ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \text{ on } \Gamma\right\},$$

$$\boldsymbol{H}_0^p(\mathrm{div},\,\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{H}^p(\mathrm{div},\,\Omega);\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\right\}.$$

$$\boldsymbol{X}_N^p(\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{X}^p(\Omega);\ \ \boldsymbol{v} \times \boldsymbol{n} = \mathbf{0} \ \ \text{on } \Gamma\right\},$$

$$\boldsymbol{X}_T^p(\Omega) = \left\{\boldsymbol{v} \in \boldsymbol{X}^p(\Omega);\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \ \ \text{on } \Gamma\right\}.$$

Properties of the functional spaces

$$\boldsymbol{K}_N^p(\Omega) = \{ \ \boldsymbol{v} \in \boldsymbol{X}_N^p(\Omega), \ \operatorname{div} \boldsymbol{v} \ = \ 0, \ \ \mathbf{curl}\, \boldsymbol{v} = \mathbf{0} \ \text{ in } \Omega \ \} .$$

$$\boldsymbol{K}_T^p(\Omega) = \{ \ \boldsymbol{v} \in \boldsymbol{X}_T^p(\Omega), \ \operatorname{div} \boldsymbol{v} \ = \ 0, \ \ \mathbf{curl}\, \boldsymbol{v} = \mathbf{0} \ \text{ in } \Omega \ \} .$$

Properties of the functional spaces

Theorem 1

The spaces $\boldsymbol{X}_N^p(\Omega)$ *and* $\boldsymbol{X}_T^p(\Omega)$ *are continuously imbedded in* $\boldsymbol{W}^{1,p}(\Omega)$.

Properties of the functional spaces

More generally, setting for m positive integer

$$\begin{aligned}\boldsymbol{X}^{m,p}(\Omega) &= \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\,\boldsymbol{v} \in W^{m-1,p}(\Omega),\\ &\quad \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{W}^{m-1,p}(\Omega),\ \boldsymbol{v}\cdot\boldsymbol{n} \in W^{m-\frac{1}{p},p}(\Gamma)\},\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{Y}^{m,p}(\Omega) &= \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\,\boldsymbol{v} \in W^{m-1,p}(\Omega),\\ &\quad \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{W}^{m-1,p}(\Omega),\ \boldsymbol{v}\times\boldsymbol{n} \in \boldsymbol{W}^{m-\frac{1}{p},p}(\Gamma)\},\end{aligned}$$

Properties of the functional spaces

Corollary 2 (Friedriech's inequalities: Amrouche-Seloula M3AS-2013)

Let $m \in \mathbb{N}^*$ and Ω of class $\mathcal{C}^{m,1}$.
i) The space $\boldsymbol{X}^{m,p}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{m,p}(\Omega)$ and we have the following estimate: for any $\boldsymbol{v}$ in $\boldsymbol{W}^{m,p}(\Omega)$,

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{m,p}(\Omega)} \leq C\Big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{W}^{m-1,p}(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{W^{m-1,p}(\Omega)} + \|\boldsymbol{v}\cdot\boldsymbol{n}\|_{W^{m-\frac{1}{p},p}(\Gamma)}\Big). \tag{1}$$

ii) The space $\boldsymbol{Y}^{m,p}(\Omega)$ is continuously imbedded in $\boldsymbol{W}^{m,p}(\Omega)$ and we have the following estimate: for any function $\boldsymbol{v}$ in $\boldsymbol{W}^{m,p}(\Omega)$,

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{m,p}(\Omega)} \leq C\Big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{W}^{m-1,p}(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|_{W^{m-1,p}(\Omega)} + \|\boldsymbol{v}\times\boldsymbol{n}\|_{\boldsymbol{W}^{m-\frac{1}{p},p}(\Gamma)}\Big). \tag{2}$$

Theorem 3 (Equivalence of norms)

i) *On the space* $\boldsymbol{X}^p_N(\Omega)$, *the seminorm*

$$\mathbf{w} \longmapsto \|\mathbf{curl}\,\mathbf{w}\|_{\mathbf{L}^p(\Omega)} + \|\mathrm{div}\,\mathbf{w}\|_{L^p(\Omega)} + \sum_{i=1}^{I} |\langle \mathbf{w}\cdot\mathbf{n}, 1\rangle_{\Gamma_i}|, \tag{3}$$

is a norm equivalent to the full norm $\|\cdot\|_{\boldsymbol{W}^{1,p}(\Omega)}$.
ii) *On the space* $\boldsymbol{X}^p_T(\Omega)$, *the seminorm*

$$\boldsymbol{w} \longmapsto \|\mathbf{curl}\ \boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} + \|\mathrm{div}\ \boldsymbol{w}\|_{\boldsymbol{L}^p(\Omega)} + \sum_{j=1}^{J} |<\boldsymbol{w}\cdot\boldsymbol{n}, 1>_{\Sigma_j}|, \tag{4}$$

is a norm equivalent to the full norm $\|\cdot\|_{\boldsymbol{W}^{1,p}(\Omega)}$.

Estimate for weak vector potential

Theorem 4

Let $\boldsymbol{F} \in \boldsymbol{L}^{6/5}(\Omega)$ *and satisfying*

$$\operatorname{div} \boldsymbol{F} = 0 \ \text{ in } \ \Omega, \quad \boldsymbol{F} \cdot \boldsymbol{n} = 0 \ \text{ on } \ \Gamma \ \text{ and } \ \langle \boldsymbol{F} \cdot \boldsymbol{n}, 1 \rangle_{\Sigma_j} = 0, \quad \forall\, 1 \leqslant j \leqslant J. \tag{5}$$

i) *There exists a unique* $\boldsymbol{v} \in \boldsymbol{W}^{1,6/5}(\Omega)$ *with* $\operatorname{div} \boldsymbol{v} = 0$ *in* Ω *such that*

$$\textbf{curl}\, \boldsymbol{v} = \boldsymbol{F} \ \text{ in } \ \Omega, \quad \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \ \text{ on } \ \Gamma \ \text{ and } \ \langle \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_i} = 0 \ \ \forall\, 1 \leqslant i \leqslant I \tag{6}$$

and satisfying the following estimate:

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,6/5}(\Omega)} \ \leqslant \ C(\Omega) \|\boldsymbol{F}\|_{\boldsymbol{L}^{6/5}(\Omega)}. \tag{7}$$

ii) *Moreover, we have*

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} \leqslant C(\Omega) \|\boldsymbol{F}\|_{[\boldsymbol{H}^1_\tau(\Omega)]'} \tag{8}$$

Stokes system with Navier type boundary conditions

$$\begin{cases} r(p) = \max\left\{1, \frac{3p}{p+3}\right\} & \text{if} \quad p \neq \frac{3}{2}, \\ r(p) > 1 & \text{if} \quad p = \frac{3}{2}. \end{cases} \tag{9}$$

Theorem 5

i) *Let* $\boldsymbol{F} \in \boldsymbol{L}^{r(p)}(\Omega)$ *with* $\operatorname{div} \boldsymbol{F} = 0$ *in* Ω *and verifying the following compatibility conditions:*

$$\text{for any} \quad \boldsymbol{v} \in \boldsymbol{K}^2_T(\Omega), \quad \int_\Omega \boldsymbol{F} \cdot \boldsymbol{v} \,\mathrm{d}\boldsymbol{x} = 0, \tag{10}$$

$$\boldsymbol{f} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma. \tag{11}$$

Then, the problem

$$(\mathcal{E}) \qquad \begin{cases} -\Delta\,\boldsymbol{\omega} = \boldsymbol{F} \quad \text{and} \quad \operatorname{div} \boldsymbol{\omega} = 0 & \text{in } \Omega, \\ \boldsymbol{\omega} \cdot \boldsymbol{n} = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{\omega} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma, \\ \langle \boldsymbol{\omega} \cdot \boldsymbol{n}, 1\rangle_{\Sigma_j} = 0, \quad 1 \leq j \leq J. \end{cases}$$

has a unique solution $\boldsymbol{\omega}$ in $\boldsymbol{W}^{1,p}(\Omega)$ satisfying the estimate:

$$\| \boldsymbol{\omega} \|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C(\Omega) \| \boldsymbol{F} \|_{\boldsymbol{L}^{r(p)}(\Omega)}. \tag{12}$$

ii) Moreover if $\boldsymbol{F} \in \boldsymbol{L}^q(\Omega)$ for $1 < q < +\infty$ and Ω is of class $\mathcal{C}^{2,1}$, then the solution $\boldsymbol{\omega}$ is in $\boldsymbol{W}^{2,q}(\Omega)$ and satisfies the estimate:

$$\|\boldsymbol{\omega}\|_{\boldsymbol{W}^{2,q}(\Omega)} \leqslant C(\Omega) \|\boldsymbol{F}\|_{\boldsymbol{L}^q(\Omega)}. \tag{13}$$

Solution for the MHD Problem

We define the following Hilbert spaces

$$
\begin{aligned}
\boldsymbol{V} &= \{\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega); \;\; \operatorname{div} \boldsymbol{u} = 0 \;\; \text{in } \Omega\}, \\
\boldsymbol{W} &= \{\boldsymbol{B} \in \boldsymbol{H}^1(\Omega); \operatorname{div} \boldsymbol{B} = 0 \text{ in } \Omega, \; \boldsymbol{B} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma, \int_{\Sigma_j} \boldsymbol{B} \cdot \boldsymbol{n} = 0, \; 1 \leqslant j \leqslant J\}, \\
\boldsymbol{Z} &= \boldsymbol{V} \times \boldsymbol{W},
\end{aligned}
$$

and we set

$$
\| \, (\boldsymbol{u}, \boldsymbol{B}) \, \|_{\boldsymbol{Z}} \; = \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{B}\|_{\boldsymbol{H}^1(\Omega)}.
$$

Solution for the MHD Problem

Theorem 6

Let $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$, $\boldsymbol{k} \in \boldsymbol{L}^{6/5}(\Omega)$ *with*

$$\operatorname{div} \boldsymbol{k} = 0 \text{ in } \Omega, \quad \boldsymbol{k} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma \quad \text{and} \quad \int_\Omega \boldsymbol{k} \cdot \boldsymbol{\varphi}\, dx = 0 \quad \forall \boldsymbol{\varphi} \in \boldsymbol{K}^2_T(\Omega) \tag{14}$$

Then Problem $(\mathcal{S})$ *has at least one weak solution*

$$(\boldsymbol{u}, \boldsymbol{B}, \pi) \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega) \times L^2(\Omega)$$

satisfying the following estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + \|\boldsymbol{B}\|_{\boldsymbol{H}^1(\Omega)} \leqslant C\left(\|\boldsymbol{f}\|_{\boldsymbol{H}^{-1}(\Omega)} + \|\boldsymbol{k}\|_{\boldsymbol{L}^{6/5}(\Omega)}\right) \tag{15}$$

Moreover, if Ω *is of class* $\mathcal{C}^{2,1}$ *then* $\boldsymbol{B} \in \boldsymbol{W}^{2,6/5}(\Omega)$.

Idea of proof:

We will use Leray-Schauder fixed point theorem to show the existence of weak solution. For proving the compactness, the idea is to apply the estimates of weak vector potential given by Theorem 4.

Regularity of solution

Theorem 7 (Regularity $W^{1,p}(\Omega)$ with $p \geq 2$)

Assume that Ω is of class $\mathcal{C}^{2,1}$. Let

$$\boldsymbol{f} \in \boldsymbol{W}^{-1,p}(\Omega) \quad \text{and} \quad \boldsymbol{k} \in \boldsymbol{L}^{r(p)}(\Omega) \quad \text{with} \quad r(p) = \frac{3p}{p+3}$$

and satisfying the condition (14). Then the weak solution of Problem $(\mathcal{S})$ given by Theorem (6) satisfies

$$(\boldsymbol{u}, \boldsymbol{B}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times \boldsymbol{W}^{2,r(p)}(\Omega) \times L^p(\Omega). \tag{16}$$

Regularity of solution

Theorem 8 (Regularity $W^{2,p}(\Omega)$ with $p \geq 6/5$)

Assume that Ω is of class $\mathcal{C}^{2,1}$. Let

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega) \quad \text{and} \quad \boldsymbol{k} \in \boldsymbol{L}^p(\Omega)$$

satisfying the condition (14). Then the weak solution of Problem $(\mathcal{S})$ given by Theorem (6) satisfies

$$(\boldsymbol{u}, \boldsymbol{B}, \pi) \in \boldsymbol{W}^{2,p}(\Omega) \times \boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega). \tag{17}$$

C. Amrouche, S. Boukassa, Existence and regularity of solution for a model in magnetohydrodynamics, *Nonlinear Analysis*, **190**, 111602, (2020).

C. Amrouche, N. Seloula, On the Stokes equation with the Navier-type boundary conditions, *Differential Equations and Applications*, **3**, 581-607, (2011).

C. Amrouche, N. Seloula, L^p-Theory for vector fields: application to the stokes equations with pressure boundary conditions, *Mathematical Models and Methods in Applied Sciences*, **23**, 37-92, (2013).

A. Bermudez, R. Munoz-Sola, R. Vazquez, Analysis of two stationary magnetohydrodynamics systems of equations including Joule heating, *Journal of Mathematical Analysis and Application*, **368**, 444-468, (2010).

K. Hamdache, B. Jaffal-Mourtada, Existence and uniqueness of solutions for the magnetohydrodynamic flow of a second grade fluid, *Mathematical Methods in the Applied Sciences*, **36**, 478-496, (2013).

C. Zhao, K. Li, On existence, uniqueness and L^r - exponential stability for stationary solutions to the MHD equations in three-dimensional domains, *ANZIAM J*, **46**, 95-109, (2004).

Thank you for your attention