Diffuse interface approach for compressible flows around moving solids of arbitrary shape

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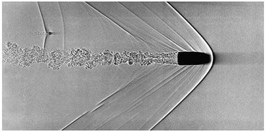
SMAI 2021, Grande Motte, France, 24/06/21

Numerical methods for multiphase flows





- **Sharp interface** methods
  - Principally based on adapted geometry
  - Ex. Lagrangian methods, tracking methods
  - Ex. Immersed/embedded boundary methods ...
- **Diffuse interface** methods
  - Principally based on the model itself and on high accurate numerical methods
  - Ex. Full Baer-Nunziato and its extension/simplifications
  - Ex. Godunov-Peshkov-Romensky model ...

Full Baer-Nunziato (BN) multiphase model

Nonconservative equation for the volume fraction function

$$\partial_t \alpha_1 + \mathbf{v}_I \cdot \nabla \alpha_1 = 0$$

First phase:

$$\partial_t (\alpha_1 \rho_1) + \nabla \cdot (\alpha_1 \rho_1 \mathbf{v}_1) = 0,$$
$$\partial_t (\alpha_1 \rho_1 \mathbf{v}_1) + \nabla \cdot (\alpha_1 (\rho_1 \mathbf{v}_1 \otimes \mathbf{v}_1 + p_1 \mathbf{I})) - p_I \nabla \alpha_1 = \alpha_1 \rho_1 \mathbf{g},$$
$$\partial_t (\alpha_1 \rho_1 E_1) + \nabla \cdot [\alpha_1 (\rho_1 E_1 + p_1) \mathbf{v}_1] - p_I \mathbf{v}_I \cdot \nabla \alpha_1 = 0$$

Second phase:

$$\partial_t (\alpha_2 \rho_2) + \nabla \cdot (\alpha_2 \rho_2 \mathbf{v}_2) = 0,$$
$$\partial_t (\alpha_2 \rho_2 \mathbf{v}_2) + \nabla \cdot (\alpha_2 (\rho_2 \mathbf{v}_2 \otimes \mathbf{v}_2 + p_2 \mathbf{I})) - p_I \nabla \alpha_2 = \alpha_2 \rho_2 \mathbf{g},$$
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Publication: M.R. Baer and J.W. Nunziato. J. Multiphase Flow (1986).

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Publication: M.R. Baer and J.W. Nunziato. J. Multiphase Flow (1986).

Reduced BN for complex nonhydrostatic free surface flows

Liquid/gas interface → complex nonhydrostatic free surface flows

Assumptions:

- Neglect the gas phase (smaller pressure fluctuations w.r.t to liquid)
- $p_I = p_{gas} = p_g$, $\mathbf{v}_I = v_{liquid} = \mathbf{v}_\ell$
- Tait EOS $p_1 = k_0 \left(\left(\frac{\rho_1}{\rho_0} \right)^\gamma - 1 \right)$

Nonconservative equation for the volume fraction function

$\partial_t \alpha + \mathbf{v}_I \cdot \nabla \alpha = 0,$

First phase:

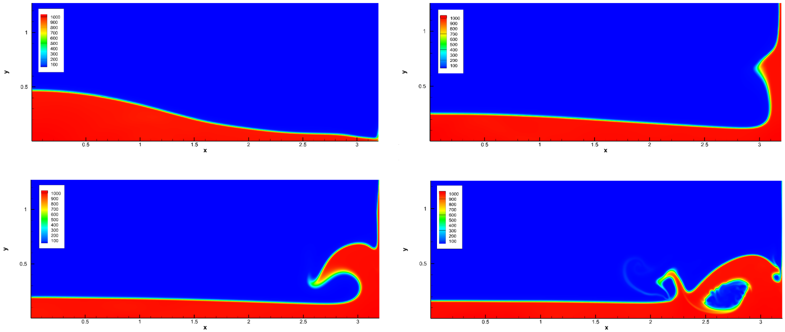
$$\partial_t (\alpha\rho) + \nabla \cdot (\alpha\rho\mathbf{v}) = 0,$$
$$\partial_t (\alpha\rho\mathbf{v}) + \nabla \cdot (\alpha (\rho\mathbf{v} \otimes \mathbf{v} + p\mathbf{I})) - p_I \nabla\alpha = \alpha\rho\mathbf{g},$$
$$\partial_t (\alpha\rho E) + \nabla \cdot [\alpha (\rho E + p) \mathbf{v}] - p_I \mathbf{v}_I \cdot \nabla\alpha = 0,$$

Publication: 📖 **E. Gaburro**, M.J. Castro, M. Dumbser, Computer & Fluids (2018).

Dambreak and impact against a vertical wall

Equilibrium IC before dambreak: $\rho(x,y) = \rho_0 \exp\left(-\frac{g\rho_0}{k_0}\left(y - y_0\left(x\right)\right)\right)$

Parameters: $k_0 = 2.62 \cdot 10^5$



Mesh: 1000×500 **Numerical method: WB FV** $\mathcal{O}(2)$

Reduced BN for compressible flows around moving solids

Assumptions:

- Neglect the solid phase because solids are assumed to be moving **rigid bodies without elastic properties**
- Consider **inviscid compressible flows**
- $p_I = p_{fluid} = p$, $\mathbf{v}_I = v_{solid} = \mathbf{v}_s$
- Stiffened gas EOS $e = \frac{p+\gamma\,\pi_k}{\rho(\gamma-1)}$

Nonconservative equation for the volume fraction function

$$\partial_t \alpha + \mathbf{v}_s \cdot \nabla \alpha = 0,$$

First phase:

$$\partial_t (\alpha\rho) + \nabla \cdot (\alpha\rho\mathbf{v}) = 0,$$

$$\partial_t (\alpha\rho\mathbf{v}) + \nabla \cdot (\alpha (\rho\mathbf{v} \otimes \mathbf{v} + p\mathbf{I})) - p_I \nabla \alpha = 0,$$

$$\partial_t (\alpha\rho E) + \nabla \cdot [\alpha (\rho E + p) \mathbf{v}] - p_I \mathbf{v}_s \cdot \nabla \alpha = 0$$

Publication: 📖 F. Kemm, **E. Gaburro**, F. Thein, M. Dumbser, Computer & Fluids (2020).

Solid and gas velocities at the material interface

We have **proved** that:

At the material **interface**,
i.e. where the volume fraction jumps from unity to zero,
the normal component of the fluid velocity assumes the value of the normal component of the solid velocity

Proved via

- either Riemann invariants
- or Generalized Rankine Hugoniot conditions (DLM theory '89)

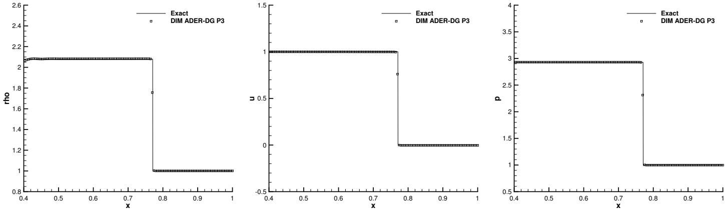
knowing that α may change from 0 to 1 only across one wave which is a contact discontinuity

$\Rightarrow$ The **non penetration boundary condition is automatically satisfied** by the model

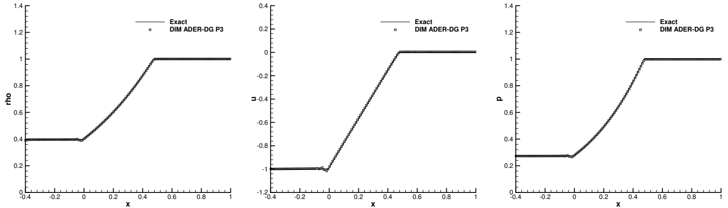
Publication: 📖 F. Kemm, **E. Gaburro**, F. Thein, M. Dumbser, Computer & Fluids (2020).

Numerical verification that fluid vel = solid vel at interfaces

✓ Solid piston **moving into** a fluid at rest causing a **right-moving shock wave**



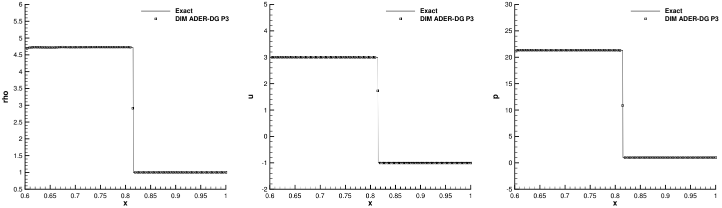
✓ Solid piston **moving away** from a fluid causing a **right-moving rarefaction**



Numerical method: DG $\mathcal{O}(4)$

Numerical verification that fluid vel = solid vel at interfaces

✓ Piston that **hits a left-moving fluid** with supersonic velocity and thus generates a **rather strong shock**, with a shock Mach number of about $M = 5$



Numerical method: DG $\mathcal{O}(4)$

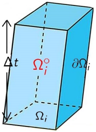
High order ADER-DG scheme

- **High order data representation in space**

$$\mathbf{u}_h(\mathbf{x}, t^n) = \mathbf{u}_h(\boldsymbol{\xi}(\mathbf{x})) = \sum_{\ell=0}^{\mathcal{N}-1} \varphi_\ell(\boldsymbol{\xi})\, \hat{\mathbf{u}}_\ell := \varphi_\ell(\boldsymbol{\xi})\, \hat{\mathbf{u}}_\ell$$

High order ADER-DG scheme

- **High order data representation in space** $\rightarrow$ $\mathbf{u}_h(\mathbf{x}, t^n)$
- **High order in time: element-local space-time ADER predictor**



$$\mathbf{q}_h(\mathbf{x}, t) = \mathbf{q}_h(\boldsymbol{\xi}(\mathbf{x}), \tau(t)) = \sum_{\ell=0}^{\mathcal{Q}-1} \theta_\ell(\boldsymbol{\xi}, \tau)\hat{\mathbf{q}}_\ell = \theta_\ell(\boldsymbol{\xi}, \tau)\hat{\mathbf{q}}_\ell$$

s.t.

$$\int_{\Omega_i^\circ} \theta_k(\mathbf{x}, t^{n+1})\mathbf{q}_h(\mathbf{x}, t^{n+1}) - \int_{\Omega_i^\circ} \theta_k(\mathbf{x}, t^n)\mathbf{u}_h(\mathbf{x}, t^n) - \int_{t^n}^{t^{n+1}} \int_{\Omega_i^\circ} \partial_t \theta_k(\mathbf{x}, t)\mathbf{q}_h(\mathbf{x}, t)$$

$$+ \int_{t^n}^{t^{n+1}} \int_{\Omega_i^\circ} \theta_k(\mathbf{x}, t)\nabla \cdot \mathbf{F}(\mathbf{q}_h(\mathbf{x}, t)) + \int_{t^n}^{t^{n+1}} \int_{\Omega_i^\circ} \theta_k(\mathbf{x}, t)\mathbf{B}(\mathbf{q}_h(\mathbf{x}, t)) \cdot \nabla \mathbf{q}_h(\mathbf{x}, t) = \mathbf{0}$$

High order ADER-DG scheme

- **High order data representation in space** $\rightarrow$ $\mathbf{u}_h(\mathbf{x}, t^n)$
- **High order in time: element-local space-time ADER predictor** $\rightarrow$ $\mathbf{q}_h(\mathbf{x}, t)$
- **Fully discrete one-step path-conservative ADER-DG scheme**

$$\left(\int_{\Omega_i} \varphi_k \varphi_l \right) \hat{\mathbf{u}}_\ell^{n+1} = \left(\int_{\Omega_i} \varphi_k \varphi_l \right) \hat{\mathbf{u}}_\ell^{n} - \int_{t^n}^{t^{n+1}} \int_{\partial\Omega_i} \varphi_k \mathcal{D}\left(\mathbf{q}_h^-, \mathbf{q}_h^+\right) \cdot \mathbf{n} + \int_{t^n}^{t^{n+1}} \int_{\Omega_i} \nabla \varphi_k \cdot \mathbf{F}(\mathbf{q}_h) - \int_{t^n}^{t^{n+1}} \int_{\Omega_i^\circ} \varphi_k \mathbf{B}(\mathbf{q}_h) \cdot \nabla \mathbf{q}_h$$

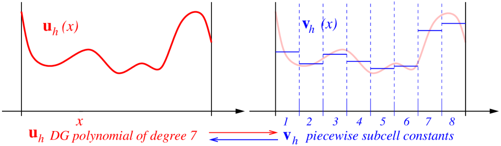
with

$$\mathcal{D}(\mathbf{q}_h^-, \mathbf{q}_h^+) \cdot \mathbf{n} = \frac{1}{2} \left(\mathbf{F}(\mathbf{q}_h^+) + \mathbf{F}(\mathbf{q}_h^-) \right) \cdot \mathbf{n} - \frac{1}{2} s_{\max} \left(\mathbf{q}_h^+ - \mathbf{q}_h^- \right) + \frac{1}{2} \left(\int_0^1 \mathbf{B}\left(\boldsymbol{\Psi}(\mathbf{q}_h^-, \mathbf{q}_h^+, \tau) \right) \cdot \mathbf{n}\, ds \right) \cdot \left(\mathbf{q}_h^+ - \mathbf{q}_h^- \right),$$

High order ADER-DG scheme

- **High order data representation in space** $\rightarrow$ $\mathbf{u}_h(\mathbf{x}, t^n)$
- **High order in time: element-local space-time ADER predictor** $\rightarrow$ $\mathbf{q}_h(\mathbf{x}, t)$
- **Fully discrete one-step path-conservative ADER-DG scheme**
- **A posteriori sub-cell Finite Volume limiter**

- Projection DG $\rightarrow$ FV
- A posteriori admissibility criteria
- FV on troubled cells
- Reconstruction FV $\rightarrow$ DG



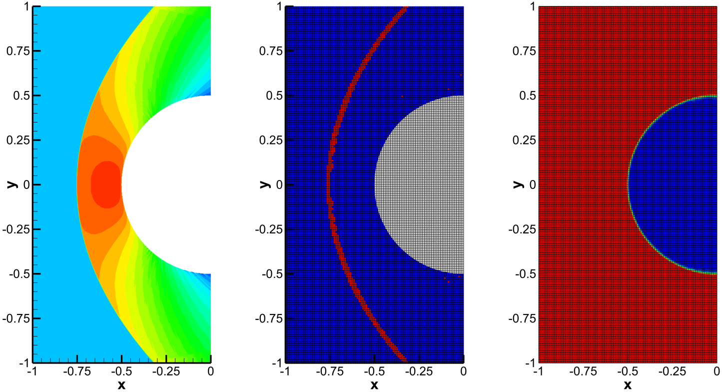
$$n_{\text{subcells}} \leq (2N+1)^d$$

because $\text{CFL}_{\text{DG}} < \frac{1}{(2N+1)^d} \rightarrow \text{CFL}_{\text{FVsubcells}} < 1$

Mach 3 flow over a blunt body

Parameters: $\epsilon = 10^{-2}$, mesh 100×200, $t_{\text{end}} = 1$

Fluid phase: $\rho = 1.4$, $u = 3$, $v = 0$, $p = 1$, $u_s = 0$, **solid velocity**: $v_s = 0$

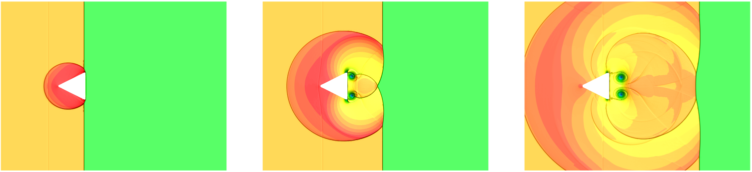


→ Note that the interface is resolved within one to two elements

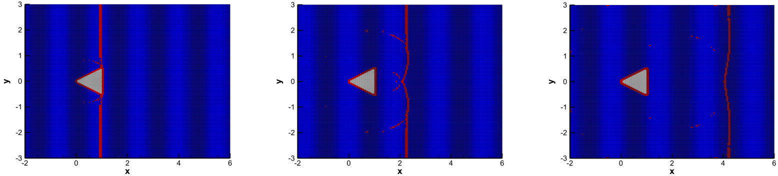
Numerical method: DG $\mathcal{O}(6)$

Flow of a shock over a wedge

Density of the compressible flows, cut off $\alpha > 0.5$



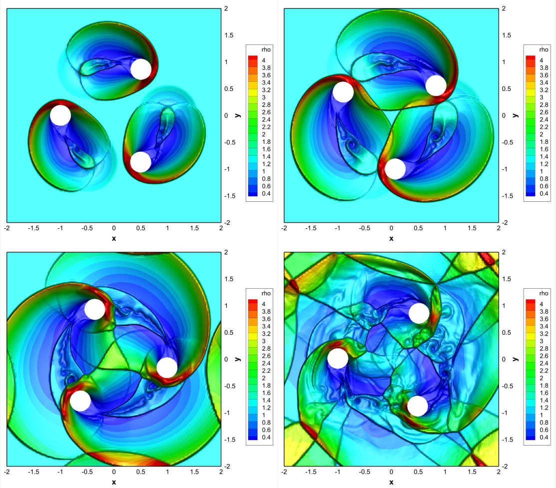
Mesh 200×150 elements, limiter activation



Numerical method: DG $\mathcal{O}(6)$

Three cylinders rotating in a compressible gas, supersonic speed

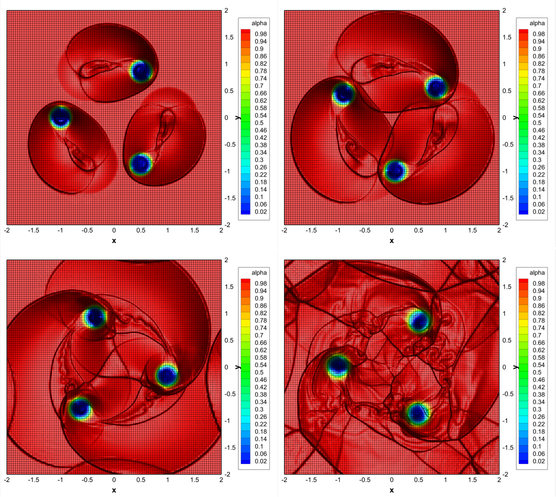
Mesh 100×100 **elements, cut off** $\alpha > 0.5$**:** ρ



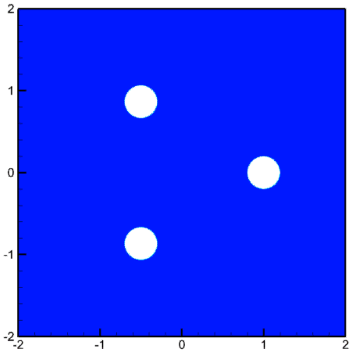
Numerical method: DG $\mathcal{O}(4)$

Three cylinders rotating in a compressible gas, supersonic speed

Mesh 100×100 **elements,** α



To be fair: a well known diffuse interface problem...



Possible **improvements**: AMR, limiter, ALE, ALE+AMR

Conclusion

Merci beaucoup pour votre attention !

A simple diffuse interface approach for compressible flows around moving solids of arbitrary shape based on a reduced Baer-Nunziato model
F. Kemm, **E. Gaburro**, F. Thein, M. Dumbser, Computers & Fluids, 2020.

A well balanced diffuse interface method for complex nonhydrostatic free surface flows
E. Gaburro, M.J. Castro, M. Dumbser, Computers & Fluids 2018.

A unified framework for the solution of hyperbolic PDE systems using high order direct Arbitrary-Lagrangian-Eulerian schemes on moving unstructured meshes with topology change
E. Gaburro, Archives of Computational Methods in Engineering (2021).

High order direct ALE schemes on moving Voronoi meshes with topology changes
E. Gaburro, W. Boscheri, S. Chiocchetti, C. Klingenberg, V. Springel, M.Dumbser, JCP (2020).

Diffuse interface with GPR model

Idea

Coupling α with the **GPR** model: a **unified model**, hyperbolic of order 1, for **continuum mechanics**

$$\frac{\partial \alpha}{\partial t} + v_k \frac{\partial \alpha}{\partial x_k} = 0$$

$$\frac{\partial(\alpha\rho)}{\partial t} + \frac{\partial(\alpha\rho v_k)}{\partial x_k} = 0$$

$$\frac{\partial(\alpha\rho v_i)}{\partial t} + \frac{\partial\left(\alpha\rho v_i v_k + \alpha p \delta_{ik} - \alpha\sigma_{ik}\right)}{\partial x_k} = \rho g_i$$

$$\frac{\partial A_{ik}}{\partial t} + \frac{\partial(A_{ij} v_j)}{\partial x_k} + v_j \left(\frac{\partial A_{ik}}{\partial x_j} - \frac{\partial A_{ij}}{\partial x_k} \right) = -\frac{1}{\theta_1(\tau_1)} E_{A_{ik}}$$

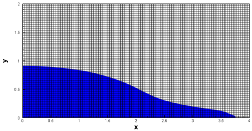
$$\frac{\partial(\alpha\rho J_i)}{\partial t} + \frac{\partial\left(\alpha\rho J_i v_k + T\delta_{ik}\right)}{\partial x_k} = -\frac{1}{\theta_2(\tau_2)} E_{J_i}$$

$$\frac{\partial(\alpha\rho S)}{\partial t} + \frac{\partial\left(\alpha\rho S v_k + E_{J_k}\right)}{\partial x_k} = \frac{\rho}{T}\left(\frac{1}{\theta_1} E_{A_{ik}} E_{A_{ik}} + \frac{1}{\theta_2} E_{J_k} E_{J_k} \right)$$

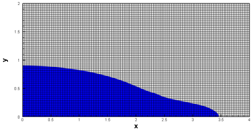
$$\frac{\partial(\alpha\rho E)}{\partial t} + \frac{\partial\left(v_k \alpha\rho E + \alpha v_i (p\delta_{ik} - \sigma_{ik})\right)}{\partial x_k} = \rho g_i v_i$$

Dambreaks

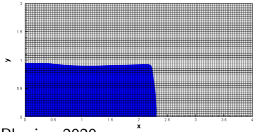
Water: low viscosity



Honey: high viscosity



Pudding: elastic solid



Publication: S. Busto, S. Chiocchetti, M. Dumbser, **E. Gaburro**, I. Peshkov, Frontiers in Physics, 2020