# On the decay rate for degenerate gradient flows subject to persistent excitation

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# Degenerate gradient flows

Consider

$$\dot{x}(t) = -c(t)c(t)^\top x(t), \qquad x \in \mathbb{R}^n,\ c : [0, +\infty) \to \mathbb{R}^n. \tag{DGF}$$

These systems appear in algorithms for, e.g.,

1. Gradient descent with incomplete knowledge of the gradient
2. Identification and model reference adaptive control
3. Consensus in multi-agent systems

**Objective**

Guarantee convergence and stability of (DGF) at the origin, and extract information on the decay rate.

# Motivation: Adaptive filters

Consider the scalar output system

$$z(t) = h^\top c(t).$$

**Problem**

Estimate the parameter $h \in \mathbb{R}^n$, knowing the input $c : \mathbb{R}_+ \to \mathbb{R}^n$ and the output $z : [0, +\infty) \to \mathbb{R}$.

Given an estimate $\hat{h} : [0, +\infty) \to \mathbb{R}^n$, we let $\hat{z}(t) = \hat{h}(t)^\top c(t)$. Then let

$$\frac{d}{dt}\hat{h}(t) = (z(t) - \hat{z}(t))c(t).$$

Then, the misalignement vector $x(t) = h - \hat{h}(t)$ satisfies (DGF):

$$\dot{x}(t) = -\left(z(t) - \hat{z}(t)\right) c(t) = -\left(x(t)^\top c(t)\right) c(t)$$

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Convergence to 0 of (DGF) $\iff$ Quality of the estimator $\hat{h}$

# Persistent excitation

We say that $c$ verifies the *persistent excitation* condition if there exists $a, b, T > 0$ such that

$$\forall t \geq 0, \qquad a\,\mathsf{Id}_n \leq \int_t^{t+T} c(s)c(s)^\top\, ds \leq b\,\mathsf{Id}_n\,. \tag{PE}$$

**Theorem (cf., Anderson, Narendra, *et al.*)**

*A signal $c$ verifies* (PE) *if and only if* (DGF) *is uniformly globally exponentially stable at* 0*. That is, there exist $C, \alpha > 0$ such that*

$$\|x(t)\| \leq Ce^{-\alpha(t-s)}\|x(s)\|, \qquad \forall t > s \geq 0.$$

- (PE) says that $c$ "visits all directions of $\mathbb{R}^n$ during a time window".
- The upper bound $b$ is *essential*. Indeed, by Barabanov *et al.* (2005), if $b = +\infty$ it can happen that

$$x(t) \longrightarrow \bar{x} \neq 0 \quad \text{as } t \to +\infty$$

# Decay rate

Under (PE), the system $\dot{x} = -cc^\top x$ is *globally exponentially stable*:

$$\|x(t)\| \leq Ce^{-\alpha t}\|x(0)\|, \qquad \forall t \geq 0. \tag{GES}$$

The *decay rate* for (DGF) is

$$R(c) := \sup\{\alpha > 0 \mid \text{(GES) holds}\} = -\limsup_{t\to+\infty} \frac{\log \|\Phi_c(t,0)\|}{t},$$

where $\Phi_c(t,0)$ is the fundamental matrix of (DGF) from 0 to $t$.

**Definition**

The *worst decay rate* is

$$R(a,b,T,n) := \inf\{R(c) \mid c \text{ satisfies (PE) with parameters } a,b,T\}.$$

$\rightsquigarrow$ Yields the *guaranteed* convergence rate of the system.

# Main result

Many lower bounds for $R(a, b, T, n)$ exist in the literature, of the type:

**Theorem (cf., Andersson and Krishnaprasad (2002))**

*There exists $C_1 > 0$ such that*

$$R(a, b, T, n) \geq \frac{C_1 a}{(1 + nb^2)T}, \qquad \forall T > 0,\ a < b,\ n \in \mathbb{N}.$$

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**Theorem (Chitour-Mason-Prandi)**

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$\rightsquigarrow$ We recover the result by Barabanov *et al.* (2005)

# Application I: Generalized persistent excitation

More general condition considered in Barabanov and Ortega (2017), Praly (2017), Efimov *et al.* (2018):

$$a_\ell \,\mathrm{Id}_n \leq \int_{\tau_\ell}^{\tau_{\ell+1}} c(s)c(s)^\top \, ds \leq b_\ell \,\mathrm{Id}_n, \qquad \forall \ell \in \mathbb{N} \tag{GPE}$$

where $a_\ell, b_\ell > 0$, and $(\tau_\ell)_{\ell\in\mathbb{N}}$ is strictly increasing with $\tau_\ell \to +\infty$.

**Theorem (Praly (2017))**

*Condition* (GPE) *guarantees global asymptotic stability of* (DGF) *if*

$$\sum_{\ell=1}^{\infty} \frac{a_\ell}{(1+b_\ell)^2} = +\infty. \tag{2}$$

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**Theorem (Chitour-Mason-Prandi)**

*For every sequence* $(a_\ell)_{\ell\in\mathbb{N}}$, $(b_\ell)_{\ell\in\mathbb{N}} \subset (0,+\infty)$ *not satisfying* (2), *there exists a signal* $c$ *satisfying* (GPE) *for which* (DGF) *is not globally asymptotically stable.*

# Application II: $L^2$-gain for (DGF) systems with linear input

Consider the controlled (DGF) system:

$$\dot{x}(t) = -c(t)c(t)^\top x(t) + u(t), \qquad u \in L^2([0,+\infty), \mathbb{R}^n).$$

Let $\gamma(c)$ be the $L^2$-gain of the input/output map $u \mapsto x$:

$$\gamma(c) = \sup_{u \in L^2 \setminus \{0\}} \frac{\|x_u\|_2}{\|u\|_2}$$

Rantzer (1999) posed the problem of determining the worst $L^2$ gain:

$$\gamma(a, b, T, n) = \sup\{\gamma(c) \mid c \text{ satisfies (PE) }\}.$$

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**Theorem (Chitour-Mason-Prandi)**

*There exists $\kappa_0, \kappa_1 > 0$ such that for all $T > 0$, $a \leq b$, $n \geq 2$, it holds*

$$\kappa_0 \frac{(1+b^2)T}{a} \leq \gamma(a,b,T,n) \leq \kappa_1 \frac{(1+nb^2)T}{a}.$$

# Sketch of the proof

**Idea**

Connect $R(a, b, T, n) = \inf R(c)$ with an optimal control problem.

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**Polar coordinates:** Letting $x = r\omega$ for $r > 0$ and $\omega \in \mathbb{S}^{n-1}$, (DGF) reads

$$\begin{cases} \dot{r} &= -(c^\top \omega)^2 r, \\ \dot{\omega} &= -c^\top \omega \left(c - (c^\top \omega)\omega\right), \end{cases} \qquad r_0 = \|x(0)\|, \quad \omega_0 = \frac{x(0)}{\|x(0)\|}.$$

- The dynamics of $\omega$ are independent of $r$.
- The dynamics of $r$ yield:

$$-\log \frac{\|x(T)\|}{\|x(0)\|} = -\log \frac{r(T)}{r(0)} = \int_0^T (c^\top \omega)^2 \, ds =: J_T(c, \omega_0).$$

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# Sketch of the proof II

Optimal control problem:

$$\mu(a, b, T, n) := \min J_T(c, \omega_0) = \min \int_0^T (c^\top \omega)^2 \, ds$$

Here, $c : [0, T] \to \mathbb{R}^n$ runs over all signals satisfying

$$a \, \mathrm{Id}_n \leq \int_0^T c(s)c(s)^\top \, ds \leq b \, \mathrm{Id}_n,$$

and $\omega$ is a solution to (Pol) with $\omega(0) = \omega_0 \in \mathbb{S}^{n-1}$.

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Steps:

1. Prove that

$$R(a,b,T,n) \leq 2 \frac{\mu(a/2, b/2, T, n)}{T}$$

$\rightsquigarrow$ Show that $\mu(a/2, b/2, T, n)$ is realized by an optimal control $c_\star : [0,T] \to \mathbb{R}^n$, which extends to a $2T$-periodic (PE) signal $c_\star : \mathbb{R}_+ \to \mathbb{R}^n$

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PMP

# More general systems

We obtain the same result for the worst rate of decay for the more general system

$$\dot{x}(t) = -S(t)x(t)$$

were $S(t) \in \mathbb{R}^{n\times n}$ is such that $S(t) \geq 0$ and for $a, b, T > 0$

$$a\,\mathrm{Id}_n \leq \int_t^{t+T} S\,ds \leq b\,\mathrm{Id}_n$$

$\rightsquigarrow$ The family of signals $S$ is obtained as the convexification of the family $cc^\top$ where $c : [0, T] \to \mathbb{R}^n$ satisfies (PE)

$\rightsquigarrow$ the worst rate of decay is realized by (DGF), e.g., $S = cc^\top$

# Perspectives

**Open question**

For $a, b, T$ fixed, what dependence on the dimension?

$$\frac{C_1}{n} \leq \lim_{b \to \infty} R(a, b, T, n) \frac{(1 + b^2)T}{a} \leq C_0.$$

- The technique used in the proof yields also the lower bound

$$R(a, b, T, n) \geq \frac{\mu(a, b, T, n)}{T}.$$

- At the moment we cannot directly access $\mu(a, b, T, n)$ for $n \neq 2$.

Thank you for your attention!

Y. Chitour, P. Mason, D. Prandi
*Worst Exponential Decay Rate for Degenerate Gradient flows subject to persistent excitation*
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