Optimal decay of the parabolic semigroup for elliptic systems with correlated coefficient fields

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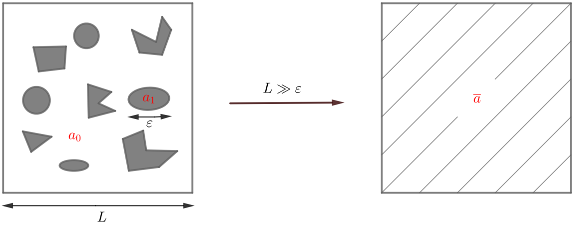
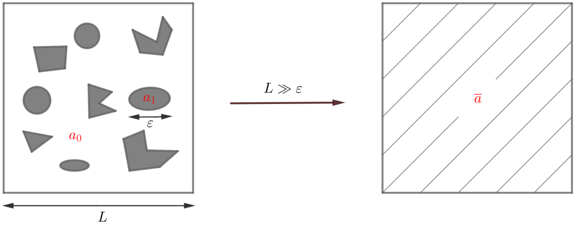
Contents

The homogenization theory: Aims at deriving the effective properties of heterogeneous systems, when heterogeneities are small compared to the characteristic size of the system.



The homogenization theory: Aims at deriving the effective properties of heterogeneous systems, when heterogeneities are small compared to the characteristic size of the system.



Mathematical models of heterogeneous media: Constitutive properties are described by functions of the form $a = a(\frac{x}{\varepsilon})$. The microstructure is **characterized statistically**, $a = a(x)$ is a realization of a random distribution.

We are interested in linear conductivity model (of thermal of electrical type):

$$-\nabla \cdot a(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon = \nabla \cdot f,$$

posed in a domain $\Omega \subset \mathbb{R}^d$ with Dirichlet boundary condition.

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$$\begin{aligned} f &= \text{exterior forcing}, \\ u_\varepsilon &= \text{temperature or electric potential}. \end{aligned}$$

Goal: Find an equivalent homogeneous model (in the limit $\varepsilon \downarrow 0$) with effective conductivity $\bar{a}$=function of the different conductivities of the microstructure and its distribution.

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The resistors are independently and identically distributed : let $(b_i)_{i\in\mathbb{N}}$ be Bernouilli's law (with parameter p) and:

$$a(\tfrac{x}{\varepsilon}) = a_0 + (a_1 - a_0)\sum_{i=1}^{+\infty} b_i \mathbb{1}_{[i-1,i]}(\tfrac{x}{\varepsilon}).$$

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We assume that $f = 0$ and we impose a difference of potential $\int_0^1 u_\varepsilon' = 1$. The conductance of the system is given by:

$$\overline{a}_\varepsilon := \left(\int_0^1 \frac{1}{a(\frac{x}{\varepsilon})}\mathrm{d}x\right)^{-1} = \left(\varepsilon\sum_{i=1}^{\frac{1}{\varepsilon}} \frac{1}{a_0 + (a_1 - a_0)b_i}\right)^{-1}.$$

Law of large numbers imply:

$$\overline{a}_\varepsilon = \left(\varepsilon \sum_{i=1}^{\frac{1}{\varepsilon}} \frac{1}{a_0 + (a_1 - a_0) b_i} \right)^{-1} \xrightarrow[\varepsilon \downarrow 0]{} \mathbb{E}\left[\frac{1}{a_0 + (a_1 - a_0) b_1} \right]^{-1}$$

$$= \left(\frac{p}{a_1} + \frac{1-p}{a_0} \right)^{-1}.$$

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Qualitative homogenization theory:

(i) There exists $\overline{u}$ s.t

$$u_\varepsilon \underset{\varepsilon \downarrow 0}{\rightarrow} \overline{u} \text{ and } \nabla u_\varepsilon \rightharpoonup \nabla \overline{u} \text{ in } \mathrm{L}^2(\Omega) \text{ a.s.}$$

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(iii) The homogenized matrix $\overline{a}$ is given by: for all $i \in [\![1, d]\!]$

$$\overline{a} e_i = \mathbb{E}[a(\nabla \phi_i + e_i)],$$

where ϕ_i denotes the corrector in the direction e_i, the distributional solution in $\mathbb{R}^d$ of

$$-\nabla \cdot a(\nabla \phi_i + e_i) = 0 \text{ with } \frac{1}{R} \fint_{\mathrm{B}_R} |\phi_i|^2 \underset{R \uparrow +\infty}{\rightarrow} 0.$$

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and the homogenization error $z_\varepsilon := u_\varepsilon^{2sc} - u_\varepsilon$ satisfies for all $\omega \subset\subset \Omega$

$$\mathbb{E}\left[\int_\omega |\nabla z_\varepsilon|^2\right] \lesssim \varepsilon^2 \mathbb{E}\left[\int_\Omega (|\phi(\tfrac{x}{\varepsilon})|^2 + |\sigma(\tfrac{x}{\varepsilon})|^2)|\nabla^2 \overline{u}(x)|^2 \mathrm{d}x\right],$$

with $\sigma = (\sigma_{ijk})_{i,j,k}$ the flux corrector satisfying

$$\nabla \cdot \sigma_i := \sum_{k=1}^{d} \partial_k \sigma_{ijk} = a(\nabla \phi_i + e_i) - \overline{a} e_i.$$

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Quantitative estimates are given by the quantification of $\mathbb{E}[|\varepsilon\phi(\frac{x}{\varepsilon})|^2] \underset{\varepsilon\downarrow 0}{\rightarrow} 0$ and $\mathbb{E}[|\varepsilon\sigma(\frac{x}{\varepsilon})|^2] \underset{\varepsilon\downarrow 0}{\rightarrow} 0$: Requires a quantification of the ergodicity.

We use a Gaussian model $a = A(g)$ for A smooth, $g : \mathbb{R}^d \to \mathbb{R}$ a Gaussian field with correlation

$$|c(x, y)| := |\mathbb{E}[g(x)g(y)]| \leq C(1 + |x - y|)^{-\beta},$$

for $C > 0$ and $\beta > 0$. The algebraic decay of the correlations **quantifies the ergodicity**.

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We use a parabolic approach:

$$\begin{cases} \partial_\tau u_i - \nabla \cdot a\nabla u_i = 0 & \text{in } \mathbb{R}^d \times (0,+\infty) \\ u(0) = \nabla \cdot ae_i \end{cases}$$

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The relationship with the corrector:

$$\phi_i = \int_0^{+\infty} u_i(\tau,\cdot) \mathrm{d}\tau.$$

We can infer optimal estimate on ϕ_i by showing optimal decay estimate of u_i.

Give more informations: The massive term approximation

$$\frac{1}{T}\phi_{T,i} - \nabla \cdot a(\nabla \phi_{T,i} + e_i) = 0,$$

can be rewritten as

$$\phi_{T,i} = \int_0^{+\infty} e^{-\frac{\tau}{T}} u_i(\tau,\cdot)\mathrm{d}\tau.$$

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Study in the small contrast regime: We assume that

$$a = (1 + A(\delta g))\mathrm{Id},$$

with $\delta \ll 1$, $A(0) = 0$ and $A'(0) = 1$. We linearize at first order $u_i \approx \delta \overline{u}_i$ with

$$\begin{cases} \partial_\tau \overline{u}_i - \Delta \overline{u}_i = 0 & \text{in } \mathbb{R}^d \times (0, +\infty) \\ \overline{u}(0) = \nabla \cdot g\, e_i \end{cases}$$

We have now an explicit formula: using the heat kernel

$$\Gamma(\tau, x) = \frac{1}{(4\pi\tau)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4\tau}},$$

one has

$$\overline{u}_i(\tau, x) = \int_{\mathbb{R}^d} \nabla\Gamma(\tau, x-y) \cdot g(y) e_i dy.$$

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Using the decay of correlations $|c(x,y)|:=|\mathbb{E}[g(x)g(y)]|\lesssim(1+|x-y|)^{-\beta}$, we bound the variance as

$$\begin{aligned}\mathbb{E}[|\overline{u}_i(\tau,0)|^2]&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\nabla\Gamma(\tau,x)\cdot e_i)(\nabla\Gamma(\tau,y)\cdot e_i)\mathbb{E}[g(x)g(y)]\mathrm{d}x\,\mathrm{d}y\\&\lesssim\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\nabla\Gamma(\tau,x)||\nabla\Gamma(\tau,y)|(1+|x-y|)^{-\beta}\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

We now consider two regimes in β:

- The weak correlation case $\beta > d$: Here, $x \mapsto (1+|x|)^{-\beta}$ is integrable, thus
$$\mathbb{E}[|\overline{u}_i(\tau,0)|^2] \lesssim \int_{\mathbb{R}^d} |\nabla\Gamma(\tau,x)|^2 \mathrm{d}x \lesssim \tau^{-1-\frac{d}{2}}.$$

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- The strong correlated case $\beta \leq d$: we first estimate

$$\begin{aligned}
&\int |\nabla\Gamma(\tau,y)|(1+|x-y|)^{-\beta}\mathrm{d}y \\
&= \underbrace{\int_{\mathbb{R}^d\setminus B_{\sqrt{\tau}}(x)} |\nabla\Gamma(\tau,y)|(1+|x-y|)^{-\beta}\mathrm{d}y}_{\leq \tau^{-\frac{\beta}{2}}\int_{\mathbb{R}^d}|\nabla\Gamma(\tau,y)|\mathrm{d}y \lesssim \tau^{-\frac{1}{2}-\frac{\beta}{2}}} \\
&+ \underbrace{\int_{B_{\sqrt{\tau}}(x)} |\nabla\Gamma(\tau,y)|(1+|x-y|)^{-\beta}\mathrm{d}y}_{\leq \sup_{y\in\mathbb{R}^d}|\nabla\Gamma(\tau,y)|\int_{B_{\sqrt{\tau}}}(1+|y|)^{-\beta}\mathrm{d}y \lesssim (1+\log(\tau)\mathbb{1}_{\beta=d})\tau^{-\frac{1}{2}-\frac{\beta}{2}}}
\end{aligned}$$

It implies:

$$\mathbb{E}[|\overline{u}_i(\tau,0)|^2] \lesssim \int_{\mathbb{R}^d} |\nabla\Gamma(\tau,x)| \int_{\mathbb{R}^d} |\nabla\Gamma(\tau,y)|(1+|x-y|)^{-\beta} dx\, dy$$
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To summarize:

$$\mathbb{E}[|\overline{u}_i(\tau,0)|^2]^{\frac{1}{2}} \lesssim \begin{cases} \tau^{-\frac{1}{2}-\frac{d}{4}} & \text{for } \beta > d, \\ \log^{\frac{1}{2}}(\tau)\tau^{-\frac{1}{2}-\frac{d}{4}} & \text{for } \beta = d, \\ \tau^{-\frac{1}{2}-\frac{\beta}{4}} & \text{for } \beta < d. \end{cases}$$

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Theorem

It holds in the non-perturbative regime and for any stochastic moments $p < +\infty$.

We deduce the control of the growth of the corrector:

Corollary (Growth of the correctors and homogenization error (under Hölder regularity of a)

Let $p < +\infty$. For all $x \in \mathbb{R}^d$,

$$
\begin{aligned}
&\mathbb{E}[|\phi(x)-\phi(0)|^p]^{\frac{1}{p}} + \mathbb{E}[|\sigma(x)-\sigma(0)|^p]^{\frac{1}{p}} \\
&\lesssim \mu(|x|) := \begin{cases} (|x|+1)^{1-\frac{\beta}{2}} & \text{for } \beta < 2, \\ \log^{\frac{1}{2}}(|x|+2) & \text{for } \beta = 2,\ d > 2 \text{ or } \beta > 2 \text{ and } d = 2, \\ \log(|x|+2) & \text{for } \beta = d = 2, \\ 1 & \text{for } \beta > 2 \text{ and } d > 2. \end{cases}
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For all $\omega \subset\subset \Omega$,

$$\mathbb{E}[\|\nabla u_\varepsilon^{2sc} - \nabla u_\varepsilon\|^p_{L^2(\omega)}]^{\frac{1}{p}} \lesssim \varepsilon\mu(\varepsilon^{-1})\left(\int \mu^2|\nabla f|^2\right)^{\frac{1}{2}}.$$

Corollary (Massive term approximation)

Define the Richardson extrapolation

$$\begin{cases} \phi^{n+1}_{T,i} = \frac{1}{2^n-1}(2^n\phi^n_{2T,i} - \phi^n_{T,i}) \\ \phi^1_{T,i} = \phi_{T,i}. \end{cases}$$

For $d \geq 2$ and $n > \frac{\max(\beta,d)}{4}$,

$$\mathbb{E}[|\nabla\phi^n_{T,i} - \nabla\phi_i|^2]^{\frac{1}{2}} \lesssim \begin{cases} T^{-\frac{\beta}{4}} & \text{for } \beta < d, \\ \log^{\frac{1}{2}}(T)\,T^{-\frac{d}{4}} & \text{for } \beta = d, \\ T^{-\frac{d}{4}} & \text{for } \beta > d. \end{cases}$$