A fast second-order discretization scheme for the linearized Green-Naghdi system with absorbing boundary conditions

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SMAI 2021

Mini-symposium :

Conditions aux limites numériques : analyse et méthodes

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Outline

1D linearized GN model

Asymptotic model for water wave

- Under the effect of the gravity, the motion of an irrotational and incompressible fluid is described by the free-surface Euler equations. Because of the complexity of this system, asymptotic models for the water wave problem were derived[a]
- The 2D Green-Naghdi model[b] includes the dispersive effects
- A one-dimensional simplification, linearized around the steady state $(H, u) := (H_0, 0) + (u_1, u_2)$, with $|(u_1, u_2)| \ll 1$, can be derived as the Green-Nagdhi (GN) system, where H is the fluid depth, u_1 the surface elevation and u_2 the velocity

[a] D. Lannes, The Water Waves Problem: Mathematical Analysis and Asymptotics, Providence, AMS, 2013.

[b] A. Green and P. Naghdi, A derivation of equations for wave propagation in water of variable depth, Journal of Fluid Mechanics, 78 (1976), pp.237-246.

Introduction

1D GN system

$$\begin{cases} (u_1)_t + (u_2)_x = 0, \\ (u_2)_t + (u_1)_x = \kappa (u_2)_{xxt}, \quad x \in \mathbb{R}, t > 0, \\ u_1(x,0) = u_1(x), u_2(x,0) = u_2(x), \quad x \in \mathbb{R}, \\ \lim\limits_{|x| \to +\infty} u_1(x,t) = 0, \lim\limits_{|x| \to +\infty} u_2(x,t) = 0, \quad t > 0. \end{cases}$$

where κ is the dispersion parameter.

Introduction

Need to truncate the unbounded domain for the numerical simulation

- Usually, the technique of absorbing/artificial boundary conditions (ABC) or the method of Perfectly Matched Layers (PML) for PDEs is used $\rightarrow$ huge literature available since more than 45 years with their pros/cons
- For the GN system, it was studied only recently for ABCs by Kazakova and Noble[a] and for PMLs by Kazakova in her PhD thesis (2018)
- Here, we are considering ABCs for the 1D GN system for a finite interval $]x_-;x_+[$

[a] M. Kazakova and P. Nobel, Discrete transparent boundary conditions for the linearized Green- Naghdi system of equations, SIAM Journal on Numerical Analysis, 1 (2020), pp.657-683.

Exact ABCs for the 1D GN system

Rewriting the system (1/2)

We consider the 1D GN problem on the whole space

$$\begin{aligned}
&\partial_t u_1(x,t) + \partial_x u_2(x,t) = 0, \\
&\partial_t u_2(x,t) + \partial_x u_1(x,t) = \kappa \partial_{xxt} u_2(x,t), && \forall\, x \in \mathbb{R},\ \forall\, t > 0, \\
&u_1(x,0) = u_1(x), u_2(x,0) = u_2(x), && \forall\, x \in \mathbb{R}, \\
&\lim_{|x| \to +\infty} u_1(x,t) = 0,\ \lim_{|x| \to +\infty} u_2(x,t) = 0, && \forall\, t > 0.
\end{aligned}$$

Introduce the new unknowns: $v_i(x,t) = e^{-\sigma t} u_i(x,t)$, $i = 1, 2$, where $\sigma > 0$ is a parameter used to later control the stability of the fast algorithm.

Rewriting the system (2/2)

Then one gets

$$
\begin{aligned}
&\partial_t v_1(x,t) + \sigma v_1(x,t) + \partial_x v_2(x,t) = 0, \\
&\partial_t v_2(x,t) + \sigma v_2(x,t) + \partial_x v_1(x,t) \\
&\qquad = \kappa \partial_{xx}(\partial_t v_2(x,t) + \sigma v_2(x,t)), && \forall\, x \in \mathbb{R},\ \forall\, t > 0, \\
&v_1(x,0) = u_1(x), v_2(x,0) = u_2(x), && \forall\, x \in \mathbb{R}, \\
&\lim_{|x|\to+\infty} v_1(x,t) = 0,\ \lim_{|x|\to+\infty} v_2(x,t) = 0, && \forall\, t > 0.
\end{aligned}
$$

For compactly supported initial data

Solve the exterior problem on $[x_+, +\infty)$ (1/2)

We obtain

$$
\begin{aligned}
&\partial_t v_1(x,t) + \sigma v_1(x,t) + \partial_x v_2(x,t) = 0,\\
&\partial_t v_2(x,t) + \sigma v_2(x,t) + \partial_x v_1(x,t)\\
&\quad = \kappa \partial_{xx}(\partial_t v_2(x,t) + \sigma v_2(x,t)), \forall\, x \in [x_+, +\infty), \forall\, t > 0,\\
&v_1(x,0) = 0, v_2(x,0) = 0, \forall\, x \in [x_+, +\infty),\\
&\lim_{x\to+\infty} v_1(x,t) = 0, \lim_{x\to+\infty} v_2(x,t) = 0, \forall\, t > 0.
\end{aligned}
$$

Exact ABCs for the 1D GN system

Solve the exterior problem on $[x_+, +\infty)$ (2/2)

- We use a relatively standard and direct method
- Laplace transform $\mathcal{L}$ the equation in time
- Solve the corresponding ODE system to extract the solution as the superposition of two waves
- Write the BC as a Dirichlet-to-Neumann map to keep the outgoing wave
- Go back to the time domain by inverse Laplace transform $\mathcal{L}^{-1}$

Exact ABCs for the 1D GN system

Truncated system with ABC in (x_-, x_+)

$$
\begin{aligned}
&\partial_t v_1(x,t) + \sigma v_1(x,t) + \partial_x v_2(x,t) = 0,\\
&\partial_t v_2(x,t) + \sigma v_2(x,t) + \partial_x v_1(x,t)\\
&\qquad = \kappa \partial_{xx}(\partial_t v_2(x,t) + \sigma v_2(x,t)), \forall x \in (x_-, x_+),\ \forall t > 0,\\
&(\mathsf{T} * v_2)(x_\pm, t) = \partial_\nu v_2(x_\pm, t), \forall t > 0,\\
&\partial_t v_1(x_\pm, t) + \sigma v_1(x_\pm, t) \pm \partial_\nu v_2(x_\pm, t) = 0,\\
&v_1(x,0) = u_1(x), v_2(x,0) = u_2(x), \forall x \in [x_-, x_+],
\end{aligned}
$$

where $\partial_\nu =$ outward normal derivative at $x_\pm$ and

$$
(\mathsf{T} * v_2)(x_\pm, t) := -\mathcal{L}^{-1}[\sqrt{\mathrm{S}(s)}\, \widehat{v}_2(x_\pm, s)](t) \quad \forall t > 0.
$$

$$
\text{with} \quad \mathrm{S}(s) := \frac{(s+\sigma)^2}{1 + \kappa (s+\sigma)^2}.
$$

Semi-discrete CN discretization of (2.1)

We have $v_i^n(x) \approx v_i(x, t_n)$, for $i = 1, 2$

$$
\begin{aligned}
&(D_\tau + \sigma E)v_1^n(x) + \partial_x E v_2^n(x) = 0, \\
&(D_\tau + \sigma E)v_2^n(x) + \partial_x E v_1^n(x) \\
&\quad = \kappa \partial_{xx}(D_\tau + \sigma E)v_2^n(x), && \forall x \in \mathbb{R},\ \forall\, n \geq 0, \\
&v_1^0(x) = u_1(x), v_2^0(x) = u_2(x), && \forall\, x \in \mathbb{R}, \\
&\lim_{|x|\to+\infty} v_1^n(x) = 0, \lim_{|x|\to+\infty} v_2^n(x) = 0, && \forall\, n \geq 1,
\end{aligned}
$$

- $\tau > 0$ is the uniform time step, $t_n = n\tau$, $0 \leq n \leq N$.
- $N\tau = T = t_N$, with $T =$ maximal time of computation.
- The operator S is: $Su = \{u^{n+1}\}_{n=0}^\infty$, for $u = \{u^n\}_{n=0}^\infty$.
- $E = (S + I)/2$ and $D_\tau = (S - I)/\tau$.
- $Su^n = (Su)^n$, $Eu^n = (Eu)^n$ and $D_\tau u^n = (D_\tau u)^n$.

ABC for this semi-discrete scheme (1/2)

How to achieve this?

- Basically you try to follow a similar path as in the time continuous case
- Replace the Laplace transform by the $\mathcal{Z}$-transform
- You can mimic the continuous approach to get ABCs

$\rightarrow$ semi-discrete version of the time convolution operator T as

$$(\mathcal{T} * v_2)^n = \sum_{j=0}^{n} \mathcal{T}_j v_2^{n-j}, \tag{3.1}$$

with the power series expansion

$$\widetilde{\mathcal{T}}(z) := -\sqrt{s(z)} = \sum_{j=0}^{\infty} \mathcal{T}_j z^j, \quad \forall\, z \in \mathbb{D}$$

$$s(z) = \mathrm{S}\left(\frac{2(1-z)}{\tau(1+z)}\right). \tag{3.2}$$

ABC for this semi-discrete scheme (2/2)

What you get: semi-discrete problem with ABC

$$
\begin{aligned}
&(D_\tau + \sigma E)v_1^n(x) + \partial_x E v_2^n(x) = 0,\\
&(D_\tau + \sigma E)v_2^n(x) + \partial_x E v_1^n(x)\\
&\quad = \kappa \partial_{xx}(D_\tau + \sigma E)v_2^n(x), \qquad \forall x \in (x_-, x_+),\ \forall\, n \geq 0,\\
&(\mathcal{T} * v_2)^n(x_\pm) = \partial_\nu v_2^n(x_\pm), \qquad \forall\, n \geq 0,\\
&(D_\tau + \sigma E)v_1^n(x_\pm) = \mp \partial_\nu v_2^n(x_\pm), \qquad \forall\, n \geq 0,\\
&v_1^0(x) = u_1(x), v_2^0(x) = u_2(x), \qquad \forall\, x \in [x_-, x_+].
\end{aligned}
\tag{3.3}
$$

Spatial discretization (1/3)

- Let M be a positive integer, $h = (x_+ - x_-)/M$ uniform mesh size. We define the mesh points:

$$x_k = x_- + (k - 1/2)\, h, \text{for } k = 0, 1, \cdots, M+1,$$
$$x_{k+1/2} = x_0 + (k + 1/2)\, h, \text{for } k = 0, 1, \cdots, M,$$

  where x_0 and x_{M+1} are two ghost points.
- In (3.3), we use $(v_2)_k^n$ to denote the numerical approximation of $v_2^n(x_k)$, with $0 \leq k \leq M+1$, and $(v_1)_k^n$ to define that of $v_1^n(x_{k-1/2})$, with $1 \leq k \leq M+1$.
- Let

$$(v_2)^n = ((v_2)_0^n, \cdots, (v_2)_{M+1}^n)$$

  and

$$(v_1)^n = ((v_1)_1^n, \cdots, (v_1)_{M+1}^n).$$

Spatial discretization (2/3)

- The linear operator which maps the $(M+2)$-dimensional vector $\omega = (\omega_0, \cdots, \omega_{M+1})$ to the M-dimensional vector $(\omega_1, \cdots, \omega_M)$ will be denoted by $\mathcal{P}$.
- Being given a vector $\chi = (\chi_1, \cdots, \chi_{M+1}) \in \mathbb{R}^{M+1}$ or $\omega = (\omega_0, \cdots, \omega_{M+1}) \in \mathbb{R}^{M+2}$, we introduce the discrete gradients $\nabla_h \chi$ and $\nabla_h w$ such that

$$\nabla_h \chi = \Big(\frac{\chi_2 - \chi_1}{h}, \frac{\chi_3 - \chi_2}{h}, \cdots, \frac{\chi_{M+1} - \chi_M}{h} \Big),$$

$$\nabla_h \omega = \Big(\frac{\omega_1 - \omega_0}{h}, \frac{\omega_2 - \omega_1}{h}, \cdots, \frac{\omega_{M+1} - \omega_M}{h} \Big),$$

respectively.

Spatial discretization (3/3)

Now, in (3.3), replacing the function $v_1^n(x)$ by the vector $v_1^n = ((v_1)_1^n, \cdots, (v_1)_{M+1}^n)$, $v_2^n(x)$ by $v_2^n = ((v_2)_0^n, \cdots, (v_2)_{M+1}^n)$ and changing $\partial_{xx} \to \triangle_h$, we obtain

the semi-discrete problem with ABC

$$\begin{aligned}
&(D_\tau + \sigma E)v_1^n + \nabla_h^n E v_2^n = 0, \\
&(D_\tau + \sigma E)\mathcal{P} v_2^n + \nabla_h E v_1^n = \kappa \triangle_h (D_\tau + \sigma E) v_2^n, \quad && \forall\, n \geq 0, \\
&\color{red}{(\mathcal{T} * \gamma^{\pm} v_2)^n - \partial_\nu^{\pm} v_2^n = 0,} && \forall\, n \geq 0,
\end{aligned}$$

$$v_1^0 = (u_1(x_{1/2}), \cdots, u_1(x_{M+1/2})),\ v_2^0 = (u_2(x_0), \cdots, u_2(x_{M+1})).$$

As it can be seen, the ABC is defined by a nonlocal time operator which is costly to evaluate. We now focus on its stable and efficient evaluation.

Rational approximation of the convolution quadrature 1/2

Padé approximation R_m of order m

The Padé approximation of the square-root function $\sqrt{s}$ on the closed right half complex plane can be written as

$$\sqrt{s} = \sqrt{1+s-1} \approx 1 + \sum_{j=1}^{m} \frac{\alpha_j (s-1)}{1+\beta_j (s-1)} \equiv R_m(s), \quad \mathrm{Re}(s) \geq 0.$$

and after some manipulations, It can be shown that:

$$R_m(s) = \lambda - \sum_{j=1}^{m} \frac{1}{g_j s + h_j}, \quad \lambda = 1 + \sum_{j=1}^{m} \alpha_j \beta_j^{-1},$$

$$h_j = \alpha_j^{-1} \beta_j (1-\beta_j), \quad g_j = \alpha_j^{-1} \beta_j^2, \quad j = 1, \cdots, m.$$

$$\alpha_j = \frac{2}{2m+1} \sin^2\left(\frac{j\pi}{2m+1}\right), \beta_j = \cos^2\left(\frac{j\pi}{2m+1}\right), \quad j = 1, \cdots, m.$$

Rational approximation of the convolution quadrature 2/2

Padé approximation R_m of order m

For all $\tau > 0$, $\sigma > 0$, and for $s(z)$ defined by (3.2), we can introduce the rational approximation $\widetilde{\mathcal{T}}^{(m)}(z)$ of the symbol $\widetilde{\mathcal{T}}(z)$ as

$$\widetilde{\mathcal{T}}^{(m)}(z) := -R_m(s(z)), \quad \forall\, m \geq 0.$$

We denote by $\mathcal{T}^{(m)}*$ the convolution operator analogously defined as (3.1) by replacing the convolution coefficients with the series expansion coefficients of the function $\widetilde{\mathcal{T}}^{(m)}(z)$.

Fast evaluation of $(\mathcal{T}^{(m)} * \gamma^{\pm} v_2)^n$

After some manipulations, we obtain

$$(\mathcal{T}^{(m)} * \gamma^{\pm} v_2)^n = \sum_{j=0}^{n} \mathcal{T}_j^{(m)} (\gamma^{\pm} v_2)^{n-j}$$

where $\mathcal{T}_j^{(m)} = \sum_{k=1}^{2m+1} C_k (\gamma_k)^j$, with

$$C_{2k-1} = \frac{A_k}{b_k + d_k}, C_{2k} = \frac{B_k}{b_k - d_k}$$

$$\gamma_{2k-1} = -\frac{a_k + c_k}{b_k + d_k}, \gamma_{2k} = -\frac{-a_k + c_k}{b_k - d_k},$$

for $1 \leq k \leq m$. (We can fix $\gamma_{2m+1} = 0$ and C_{2m+1}.)

$\rightarrow$ we have a fast implementation with $\mathcal{O}(mn)$ operations.

Adaptive error control of the fast evaluation with m

Let us assume that the condition $\sigma \geq \frac{1}{\sqrt{2\kappa}}$ is satisfied, the time step τ is small enough and m is sufficiently large, i.e. it fulfills

$$2m+1 \geq \frac{\ln \epsilon}{\ln(1-\delta)}, \quad \text{for some } \epsilon \in \left(0, \frac{\mu\sqrt{\kappa}\tau^3}{8}\right],$$

with $\mu(\kappa,\sigma) = \frac{\sqrt{2}\sigma}{2\sqrt{1+\kappa\sigma^2}}$ and

$$\delta(\kappa,\sigma) = \frac{\sqrt{2}\sigma\sqrt{1+\kappa\sigma^2}}{\sigma^2+\sqrt{2}\sigma\sqrt{1+\kappa\sigma^2}+1+\kappa\sigma^2}. \tag{4.1}$$

Then, the following inequality holds

$$\max_{z\in\partial\mathbb{D}} |\widetilde{\mathcal{T}}^{(m)}(z) - \widetilde{\mathcal{T}}(z)| \leq \mu\frac{\tau^3}{2}.$$

In addition, the full scheme can be proved to be 2nd order accurate both in space and time (too long).

Configuration

- We take $\sigma = 1/\sqrt{2\kappa}$ and adapt the number of Padé expansion terms following the rule (with μ and δ given by (4.1))

$$m = \frac{\ln \epsilon}{2 \ln (1 - \delta)}, \quad \epsilon = \frac{\mu \sqrt{\kappa} \tau^3}{8}.$$

  For N fixed (with $N\tau = T$), the total computational cost to efficiently evaluate the convolution is $\mathcal{O}(mN) = \mathcal{O}(N \log(N))$.
- The initial distribution for the free-surface elevation is

$$u_1(x, 0) = \exp(-400(x - 0.5)) \sin(20\pi x),$$

  and set $u_2(x, 0) = 0$. The data u_1 can be considered as compactly supported in $[x_-, x_+] = [0, 1]$.
- We fix $\kappa = 10^{-3}$, $T = 1$ and $N = M = 1280$.

Numerical results 1/4

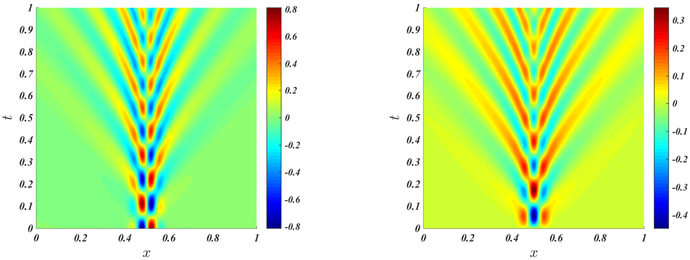

Figure: Left: surface elevation u_1; Right: velocity u_2

Numerical results 2/4

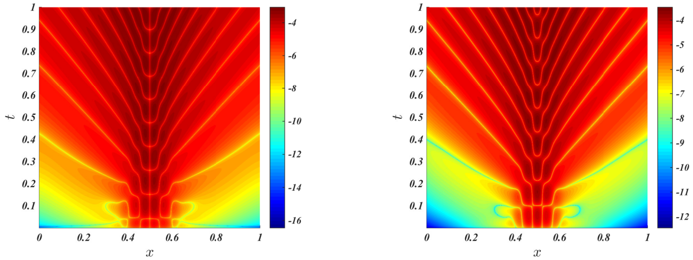

Figure: Error $\log_{10}(|u_j^{\text{ref}} - u_j|)$ for $j = 1, 2$

Numerical results 3/4

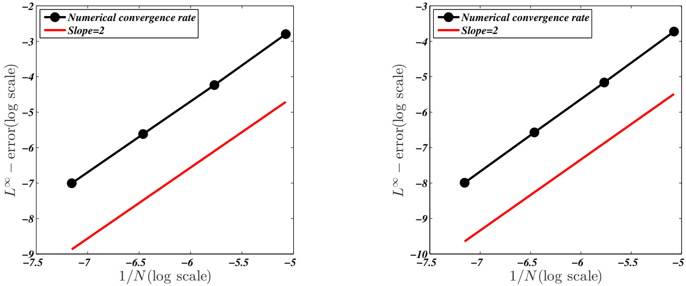

Figure: $\max_{x\in[0,1]} |u_j(x,T) - u_j^{\text{ref}}(x,T)|$ vs. N (log scale), for $j = 1, 2$

Numerical results 4/4

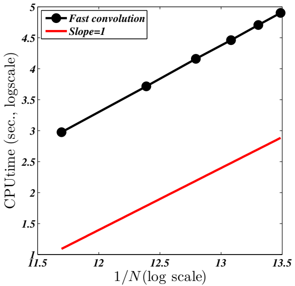

Figure: Computational time for the evaluation of the convolution by the fast algorithm vs N (for $M = 160$). The total number of time steps N increases from $N = 1.2 \times 10^5$ to $N = 7.2 \times 10^5$, with step 1.2×10^5. We observe a slope equal to 1, showing that the cost is linear according to $\log(1/N)$, i.e. as $\mathcal{O}(N \log N)$ for the computational time.

Conclusion - perspectives

- Solution of the 1D GN system with ABCs, efficient algorithm, second-order scheme*
- A remaining problem is that the damping term $e^{-\sigma t}$ should satisfies the stability condition $\sigma \geq \frac{1}{\sqrt{2\kappa}}$. For a small dispersion κ, the damping term $e^{-\sigma t}$ which decays too fast may bring some numerical errors.
- Extensions to higher-dimensional problems still need further investigations.
- The variable coefficients and nonlinear cases of the Green-Naghdi system remain open problems as well as the case of the two-layer Green-Naghdi system.
- and **thank you for your attention**

*G. Pang, S. Ji and X. Antoine, A fast second-order discretization scheme for the linearized Green-Naghdi system with absorbing boundary conditions, HAL Id : hal-03130074, submitted 2021.