When Lyapunov meets Poincaré and (log-)Sobolev

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based on joint works with
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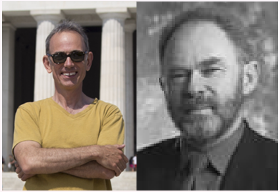
Henri Poincaré (1854-1912)

Serguei Sobolev (1908-1989)

Alexandre Liapunov (1857-1918)

Henri Poincaré

Leonard Gross



Sean Meyn and Richard Tweedie

Poincaré inequality: $\Omega \subset \mathbb{R}^n$ bounded open, f smooth with $f = 0$ on $\partial\Omega$

$$\int_\Omega |f|^2 dx \leq C \int_\Omega |\nabla f|^2 dx$$

Sobolev Inequality : $f : \mathbb{R}^n \to \mathbb{R}$ smooth compactly supp.

$$\|f\|^2_{\frac{2n}{n-2}} \leq C_n \int_{\mathbb{R}^n} |\nabla f|^2 dx$$

Poincaré inequality

$\Omega \subset \mathbb{R}^n$ bounded open, f smooth with $f = 0$ on $\partial\Omega$

$$\int_\Omega |f|^2 dx \leq C \int_\Omega |\nabla f|^2 dx$$

useful in PDE theory to solve Poisson equation

$$\begin{cases} -\Delta v = g & \text{on } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

by asserting that

$$\langle u, v \rangle = \int_\Omega \nabla u \cdot \nabla v \, dx$$

is equivalent to

$$\langle u, v \rangle_{H^1} = \int_\Omega (\nabla U \cdot \nabla v + uv) \, dx$$

but in fact this inequality is probably due to Neumann or Schwarz...

Poincaré(-Wirtinger) inequality

We will be more interested in the following: let Ω be an open, bounded, regular and convex subspace of $\mathbb{R}^n$, and f smooth then

$$Var(f) := \int_\Omega \left(f - \int_\Omega f dx \right)^2 dx \leq C(\Omega) \int_\Omega |\nabla f|^2 dx$$

useful for the spectral problem.

Indeed, find k_j, u_j such that

$$\begin{cases} -\Delta u_j = k_j u_j & \text{on } \Omega \\ \frac{\partial u_j}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

By multiplying by u_j, and integration by parts, one has

$$k_j = \frac{\int_\Omega |\nabla u_j|^2 dx}{\int_\Omega u_j^2 dx}$$

so that Poincaré inequality gives a bound on k_2 ($k_1 = 0$ for constant function), and recursively by restricting functions for every k_j.

The proof of Poincaré was quite ingenious using for the first time duplication: let $\int_\Omega f\, d\mu = 0$

$$Var(f) = \frac{1}{2|\Omega|}\int_\Omega\int_\Omega (f(x) - f(x'))^2 dxdx'$$

and by convexity of the domain

$$\begin{aligned} |f(x) - f(x')|^2 &= \left|\int_0^1 (x - x')\cdot\nabla f(tx + (1-t)x')dt\right|^2 \\ &\leq diam(\Omega)^2 \int_0^1 |\nabla f(tx + (1-t)x')|^2 dt \end{aligned}$$

by Cauchy-Schwartz. The proofs ends by a change of variable argument.

For the best constant see Payne-Weinberger ($C = diam(\Omega)^2/\pi^2$).

A more general framework

For simplicity, μ is a probability measure with potential V:
$d\mu = e^{-V(x)}dx, \qquad \mathbf{L} = \Delta - \nabla V.\nabla$
and the natural diffusion process with generator $\mathbf{L}$ is

$$dX_t = \sqrt{2}dB_t - \nabla V(X_t)dt$$

whose associated Markov semigroup is denoted P_t (reversible wrt μ).

We say that μ satisfies a Poincaré inequality if for all smooth functions

$$Var_\mu(f) = \int f^2 d\mu - \left(\int f d\mu\right)^2 \leq C \int -f\mathbf{L}f\, d\mu.$$

Remark that $\int |\nabla f|^2 d\mu = \int -f\mathbf{L}f\, d\mu$ and that $-\mathbf{L}$ is a positive operator, and the inequality gives also a lower bound on the spectrum of $-\mathbf{L}$, and thus is also called spectral gap.

But it also has many interesting consequences, which have triggered the interest for the inequality and the evaluation of its Poincaré constant.

Long time behaviour

A Poincaré inequality with constant C is equivalent to

$$\|P_t f - \mu(f)\|^2 \leq e^{-2t/C}\, Var_\mu(f)$$

Very useful for algorithms (Langevin, MALA,...)...

Proof: take $\mu(f) = 0$

$$\frac{d}{dt}\int (P_t f)^2 d\mu = 2\int P_t f \mathbf{L} P_t f d\mu \leq -\frac{2}{C}\int (P_t f)^2 d\mu$$

and "Gronwall's lemma". The other implication is even simpler.

Long time behaviour

A Poincaré inequality with constant C is equivalent to

$$\|P_t f - \mu(f)\|^2 \leq e^{-2t/C}\, Var_\mu(f)$$

Tensorization

If μ satisfies a Poincaré of constant C so does $\mu^{\otimes n}$ with constant C

Adimensionnal properties... Statistical mechanics...

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$$\|P_t f - \mu(f)\|^2 \leq e^{-2t/C}\, Var_\mu(f)$$

Tensorization

If μ satisfies a Poincaré of constant C so does $\mu^{\otimes n}$ with constant C

Concentration (Gromov-Milman)

If μ satisfies a Poincaré of constant C, then if $\delta < 2/\sqrt{C}$,

$$\mu(e^{\delta |x|}) < \infty$$

Useful for quantitative law of large numbers : $X_i \overset{i.i.d.}{\sim} \mu$ which satisfies PI then for all $1-$lipschitzian function f

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} f(X_i) - \mu(f) \geq r\right) \leq e^{-nK\min(r^2,r)}.$$

How to prove a Poincaré inequality?

Consider the Gaussian case: $d\gamma = Z^{-1}e^{-|x|^2/2}dx$, $\mathbf{L} = \Delta - x.\nabla$ and

$$P_t f(x) = \int f(e^{-t}x + \sqrt{1-e^{-2t}})d\gamma(y).$$

By integration by part, Cauchy-Schwarz and μ-invariance of P_t

$$\begin{aligned} Var_\gamma(f) &= -\int_0^\infty \frac{d}{dt}(P_t f)^2 d\gamma \\ &= 2\int_0^\infty \int |\nabla P_t f|^2 d\gamma \\ &= 2\int \int_0^\infty e^{-2t}|\nabla f|^2 dt d\gamma \\ &\leq \int |\nabla f|^2 d\gamma. \end{aligned}$$

Remark that everything works if

$$|\nabla P_t f|^2 \leq e^{-t/C} |\nabla P_t f|^2.$$

It is (roughly) the approach by curvature-dimension and Γ_2 calculus of Bakry-Emery, which works if $Hess(V) \geq C^{-1} Id > 0$.

Otherwise, there is

- Hardy-Muckenhoupt criterion in dimension 1.
- perturbation argument starting from a known inequality (Holley-Stroock, Cattiaux-G.).
- true for every V convex (Bobkov)
- ingenious works on particular cases
- and a method we'll see later

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(log-)Sobolev inequality

Sobolev Inequality : $f : \mathbb{R}^n \to \mathbb{R}$ smooth compactly supp., $n > 2$

$$\|f\|^2_{\frac{2n}{n-2}} \leq C_n \int_{\mathbb{R}^n} |\nabla f|^2 dx$$

powerful on compact embeddings of Sobolev spaces.

A consequence on Gaussian : assume first that $\int f^2 dx = 1$, then by Jensen's inequality (with $p = \frac{2n}{n-2}$)

$$\begin{aligned} \log\left(C_n \int_{\mathbb{R}^n} |\nabla f|^2 dx\right) &\geq \frac{2}{p} \log\left(\int_{\mathbb{R}^n} |f|^{p-2} f^2 \, dx\right) \\ &\geq \frac{p-2}{p} \int_{\mathbb{R}^n} f^2 \log(f^2) \, dx \end{aligned}$$

which is a form of logarithmic Sobolev inequality.

There is an issue on sharp constants: apply it to $f^{\otimes kn}$ and let $k \to \infty$ then

$$\int_{\mathbb{R}^n} f^2 \log(f^2)\, dx \leq \frac{n}{2} \log\left(\frac{2}{n\pi e} \int_{\mathbb{R}^n} |\nabla f|^2 dx\right)$$

which is the sharp Euclidean logarithmic Sobolev inequality.

Now change f^2 into $f^2 e^{-|x|^2/2}$ with $\int f^2 d\gamma = 1$, to get

$$\int_{\mathbb{R}^n} f^2 \log(f^2)\, d\gamma \leq \int_{\mathbb{R}^n} |\nabla f|^2 d\gamma$$

the Gaussian logarithmic Sobolev inequality (L. Gross in 1975).

But there are at least 15 different proofs... and in particular a modification of our proof in the Poincaré case still works, requiring another commutation

$$|\nabla P_t f| \leq e^{-ct} P_t |\nabla f|$$

It has lead to the Γ_2 calculus method of Bakry-Emery, based on Bochner inequality and curvature-dimension condition, and then extended to general spaces by Lott-Sturm-Villani, Bakry-Ledoux, Ambrosio-Gigli-Savare, Wang, Kuwada, Bolley-Gentil-G., ...

A more general framework

For simplicity, we will consider the case where μ is a probability measure with potential V:

$$d\mu = e^{-V(x)}dx, \qquad \mathbf{L} = \Delta - \nabla V.\nabla$$

and the natural diffusion process generated by

$$dX_t = \sqrt{2}dB_t - \nabla V(X_t)dt$$

whose associated semigroup is denoted P_t.

We say that μ satisfies a logarithmic Sobolev inequality (LSI) if for all smooth functions with

$$Ent_\mu(f) = \int f^2 \log\left(\frac{f^2}{\int f^2 d\mu}\right) \leq C \int -f\mathbf{L}f \, d\mu.$$

Consequences of LSI

Long time behaviour

A LSI with constant C is equivalent to

$$Ent_\mu(P_t f) \leq e^{-t/C}\, Ent_\mu(f)$$

Still very useful for algorithmic applications,... same proof than for Poincaré inequality.

Long time behaviour

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Tensorization

If μ satisfies LSI of constant C so does $\mu^{\otimes n}$ with constant C

Concentration (Herbst argument)

If μ satisfies a logarithmic Sobolev ineq. of constant C, then if $\delta < 2/C$,

$$\mu(e^{\delta|x|^2}) < \infty$$

If $X_i \overset{i.i.d.}{\sim} \mu$ which satisfies a LSI then for every $1-$lipschitzian f

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} f(X_i) - \mu(f) \geq r\right) \leq e^{-nKr^2}.$$

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Hypercontractivity (Nelson, Gross)

A logarithmic Sobolev inequality with constant C is equivalent to $\forall p > 1$

$$\|P_t f\|_{1+(p-1)e^{2t/C}} \leq \|f\|_p$$

How to prove a logarithmic Sobolev inequality?

- Γ_2 calculus of Bakry-Emery (i.e $Hess(V) \geq \rho Id > 0$).
- geometric convexity (Prekopa-Leindler),
- transportation method (Cordero-Erausquin, Mc Cann,...),
- generalized Hardy-Muckenhoupt criterion in dimension 1.
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One issue : what of a probabilistic characterization, i.e. trajectorial, of Poincaré and logarithmic Sobolev inequality?

 will come into play!!!

Dynamical system $\dot{x}_t = f(x_t)$, if for $W > 0$ around 0 and

$$\dot{V}(x_t) < 0$$

then the equilibrium is asymptotically stable (equivalence).

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Lyapunov method

Adaptation of the Lyapunov method by Meyn-Tweedie:
let $\mathbf{L} = \Delta - \nabla V.\nabla$ be the generator symmetric wrt $d\mu = e^{-V}dx$.

Lyapunov condition : $\mathbf{L}W \leq -\lambda W + b1_C$ for some $W \geq 1$ and set C.

This condition is not "hard" to verify : think of the Gaussian case...

$$L = \Delta - x.\nabla$$

Choose

$$W(x) = 1 + |x|^2/2 \qquad \mathbf{L}W(x) = d - |x|^2$$

or

$$W(x) = e^{a|x|^2/2} \qquad \mathbf{L}W(x) = (ad - (a - a^2)|x|^2)W(x).$$

Probabilistic approach: coupling

We have that a Lyapunov condition is equivalent to

$$\forall x, \qquad \mathbb{E}_x\left(e^{\lambda T_C}\right) \le V(x)$$

where $T_C = \inf\{t > 0;\, X_t \in C\}$.
Indeed by Itô's formula

$$\begin{aligned}\mathbb{E}_x(e^{\lambda \wedge T_C} W(X_{t\wedge T_C})) &= W(x) + \mathbb{E}_x\left(\int_0^{t\wedge T_C} (\mathbf{L}W(X_s) - \lambda W(X_s) ds\right)\\ &\le W(x) + \mathbb{E}_x\left(\int_0^{t\wedge T_C} b1_C(X_s) ds\right)\end{aligned}$$

The second condition for Meyn-Tweedie's approach:

$$\text{minorization condition : } \forall x \in C,\ P^*_{t_0}(x,\cdot) \geq \varepsilon\nu(\cdot)$$

which may be read as

$$P^*_{t_0}(x,\cdot) = \varepsilon\nu(\cdot) + (1-\varepsilon)\frac{P^*_{t_0}(x,\cdot) - \varepsilon\nu(\cdot)}{1-\varepsilon}$$

One then gets by coupling that

$$\|\mathcal{L}(X_t^x) - \mu\|_{TV} \leq mW(x)\, e^{-\alpha t}$$

however not so quantitative (due to minorization condition), but equivalent to a Lyapunov condition.

Lyapunov meets Poincaré

How to prove Poincaré or logarithmic Sobolev inequality from Lyapunov?
Let's start with Poincaré :

$$
\begin{aligned}
Var_\mu(f) &\leq \int (f - m_C)^2 d\mu \\
&\leq \frac{1}{\lambda}\int \frac{-\mathbf{L}W}{W}(f - m_c)^2 d\mu + \frac{b}{\lambda}\int_C (f - m_C)^2 d\mu
\end{aligned}
$$

Take then m_C the mean of f wrt μ restricted to C and a local Poincaré inequality

$$
\int_C \left(f - \int_C f d\mu \right)^2 d\mu \leq \kappa_C \int_C |\nabla f|^2 d\mu
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for the second term (by perturbation from the original Poincaré-Wirtinger inequality).

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$$\int_C \left(f - \int_C f d\mu\right)^2 d\mu \le \kappa_C \int_C |\nabla f|^2 d\mu$$

for the second term (by perturbation from the original Poincaré-Wirtinger inequality).

For the first term, use a simple calculus

$$\int \frac{-\mathbf{L}W}{W} f^2 d\mu = \int \left\langle \nabla \left(\frac{f^2}{W} \right), \nabla W \right\rangle d\mu$$

$$= 2 \int \frac{f}{W} \langle \nabla f, \nabla W \rangle d\mu - \int \frac{f^2}{W^2} |\nabla W|^2 d\mu$$

$$= - \int \left| \frac{f}{W} \nabla W - \nabla f \right|^2 d\mu + \int |\nabla f|^2 d\mu$$

$$\leq \int |\nabla f|^2 d\mu.$$

So that by Lyapunov condition and local Poincaré inequality

$$Var_\mu(f) \leq \frac{1}{\lambda}(1 + b\,\kappa_C) \int |\nabla f|^2 d\mu$$

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$$\leq \int |\nabla f|^2 d\mu.$$

So that by Lyapunov condition and local Poincaré inequality

$$Var_\mu(f) \leq \frac{1}{\lambda}(1 + b\,\kappa_C) \int |\nabla f|^2 d\mu$$

Probabilistic form of Poincaré inequality

In fact, by using concentration of Markov functionals argument we can prove the reverse statement so that

Theorem

A Poincaré inequality is equivalent to Lyapunov condition

$$\mathbf{L}W \leq -\lambda W + b1_C$$

and equivalent to the existence of a nice set U so that for some $\delta > 0$

$$\forall x, \qquad \mathbb{E}_x\left(e^{\delta T_U}\right) < \infty$$

where $T_U = \inf\{t \geq 0;\ X_t \in U\}$.

Probabilistic form of logarithmic Sobolev inequality

We may generalize this to LSI. Suppose that for some $a > 0$, $\mu(e^{aV}) < \infty$.

Theorem

A logarithmic Sobolev inequality for $d\mu = e^{-V}dx$ is equivalent to reinforced Lyapunov condition

$$\mathbf{L}W(x) \leq -\lambda V(x)\, W(x) + b$$

and equivalent to the existence of a nice set U so that for some $\delta > 0$

$$\forall x, \qquad \mathbb{E}_x\left(e^{\delta \int_0^{T_U} V(X_s)ds}\right) < \infty$$

where $T_U = \inf\{t \geq 0;\, X_t \in U\}$.

Scheme of Proof:

1. Lyapunov $\Longrightarrow$ LSI : same type argument than for Poincaré establishing a form of Super-Poincaré inequality (Nash form of LSI)
2. LSI of constant C $\Longrightarrow$ reinforced Lyapunov : let

$$\rho \leq 1/(2C),\, b = 2\mu(e^{aV}),\, \varphi = \rho(-aV + 2\mu(e^{aV})$$

denote $Hu = -\mathbf{L}u + \varphi u$. One gets using LSI

$$\frac{1}{2}\left(\mu(-u\mathbf{L}u) + \rho b\mu(u^2)\right) \leq \mu(u\,Hu) \leq \mu(-u\mathbf{L}u) + \rho b\mu(u^2)$$

and apply Lax-Milgram theory to the coercive form $\mu(u\,Hv)$ and the maximum principle.

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and apply Lax-Milgram theory to the coercive form $\mu(u\,Hv)$ and the maximum principle.

Final comments and open problems

We would have so much to say on this topic... let's focus on the Lyapunov method.

Comments:

- Lyapunov method for functional inequalities is easy, but furnishes rarely sharp constant.
- descent from infinity and ultracontractivity can also be studied by Lyapunov techniques.
- Lyapunov conditions can be adpated to weak and super-Poincaré inequalities.
- Lyapunov techniques can imply directly concentration result, via transportation inequalities.

Partially open problems:

- Lyapunov method in the discrete time case, for logarithmic Sobolev inequality for example.
- acceleration of convergence and non symmetric case (hypocoercivity).
- Sharp constants (KLS conjecture,...)

Thank you for your attention!