DE LA RECHERCHE À L'INDUSTRIE

Moment based reduced models for uncertain hyperbolic systems

SMAI 2021

Gaël Poëtte CEA/DAM/DIF → CEA/CESTA/DAM
Didier Lucor IJLRA/CNRS → LISN/CNRS
Bruno Després CEA/DAM/DIF → LJLL

La Grande Motte

22 juin 2021

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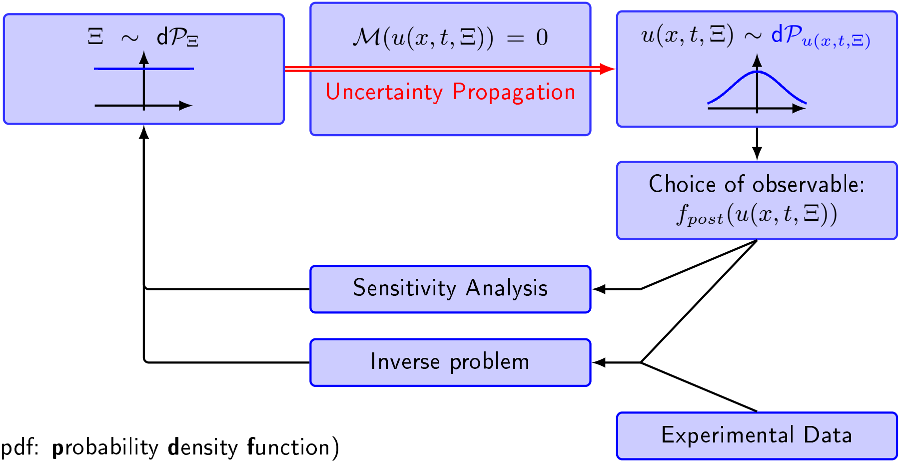
Few words on Uncertainty Analysis
An important tool for decision making



(pdf: **p**robability **d**ensity **f**unction)

- The models through which we want to propagate uncertainties:

$$\left\{\begin{array}{ll} \partial_t u(x,t) + \partial_x f(u(x,t)) = 0, & x \in \mathbb{R}, t > 0, \\ u(x,t=0) = u_0(x), & x \in \mathbb{R}, \\ u \in \mathbb{R}^n, f(u) \in \mathbb{R}^n (smooth). & \end{array}\right.$$

Références : [24, 25]

- Particularity: existence of discontinuous solutions.

Hyperbolicity (wellposedness)

The system is hyperbolic (well-posed) if $\forall u \in \mathcal{U}$

- $A(u) = \nabla_u f(u)$ is diagonalizable in $\mathbb{R}^n$ in a complete basis of eigenvectors.
- If $\exists$ a strictly convex mathematical entropy (s, g):

$$\nabla^2_{u,u} s(u) > 0 \quad \text{and} \quad \partial_t s(u) + \partial_x g(u) \leq 0.$$

In this presentation, we consider systems owning such mathematical entropy.

A simple uncertain hydrodynamical problem ('fil rouge 1')
Model $\mathcal{M}$ and its behaviour

- Compressible gas dynamics, 1D cartesian coordinates:

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho v) + \partial_x(\rho v^2 + p) = 0, \\ \partial_t(\rho e) + \partial_x(\rho v e + p v) = 0, \end{cases}$$

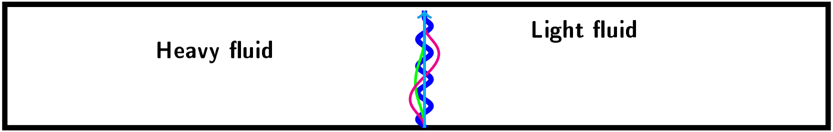
with a perfect gas closure: $p = (\gamma - 1)\left(\rho e - \frac{(\rho v)^2}{2\rho}\right)$.

- Uncertain Sod shock tube or uncertain Riemann problem

$x_{\text{int}}(t = 0, \Xi) = 0.5 + 0.05\Xi, \Xi \in [-1, 1]$ parametrizes a uniform law,

final time $t = 0.14$, $\gamma = 1.4$.

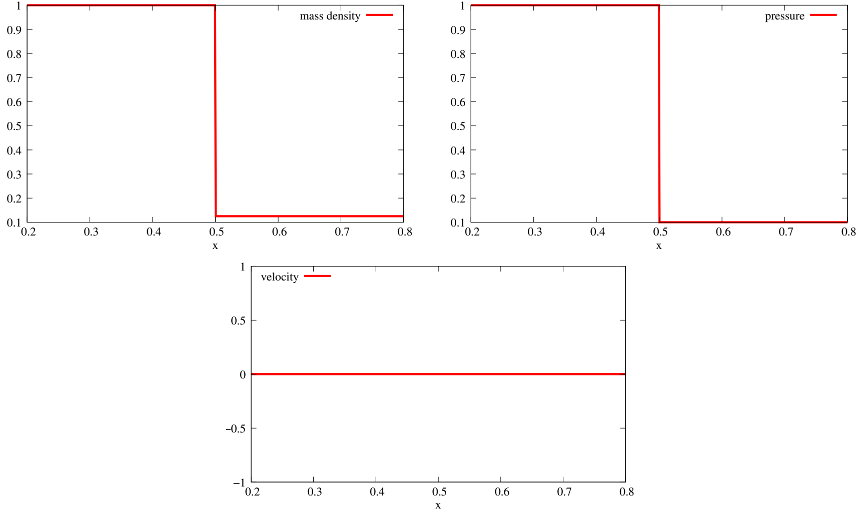
Uncertainties



$x_{\text{int}}(t = 0, \Xi)$

A simple hydrodynamical example ('fil rouge 1')

Example for one realisation of the interface position (deterministic)



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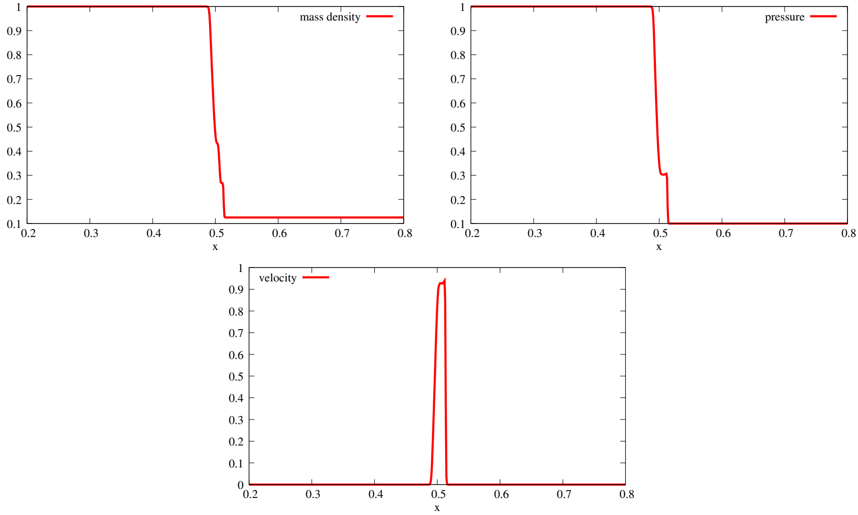

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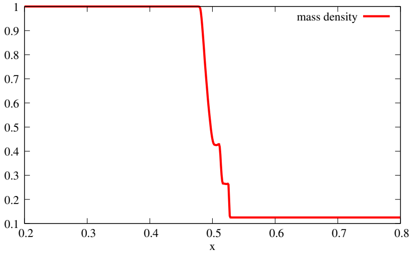

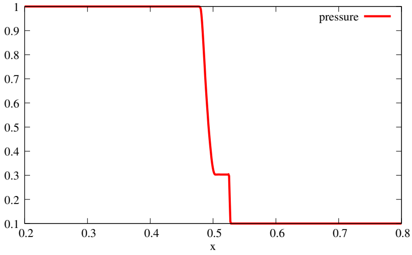

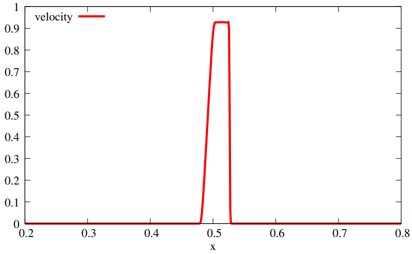

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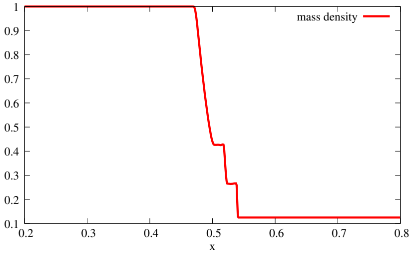

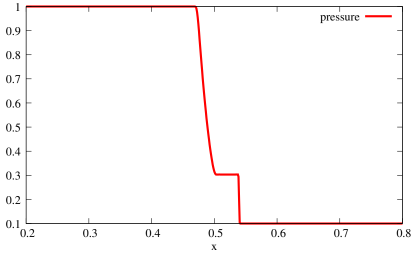

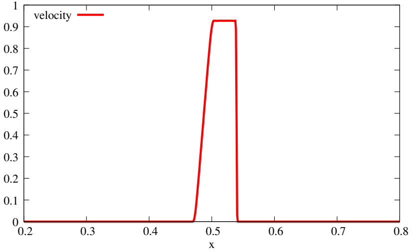

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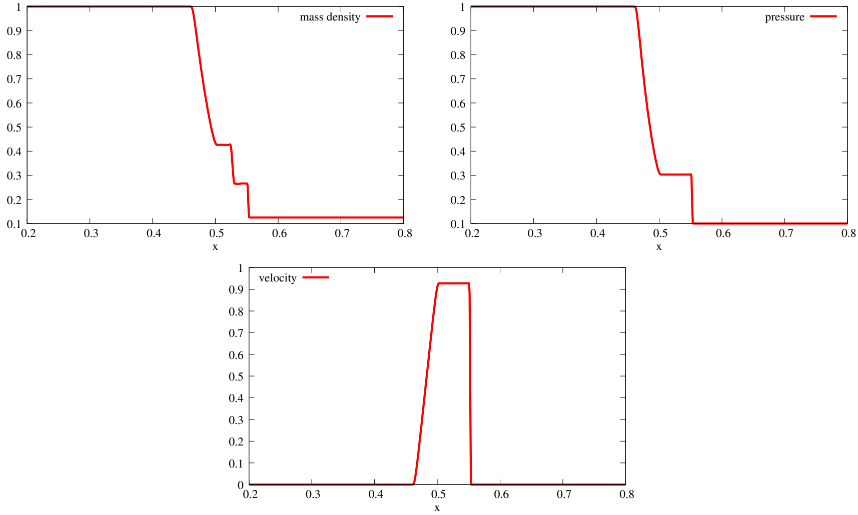

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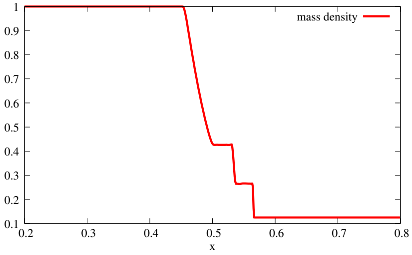

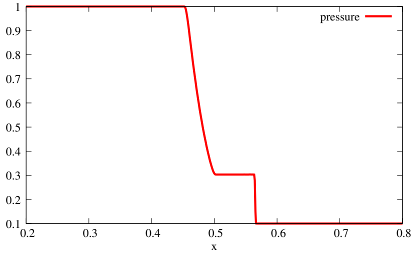

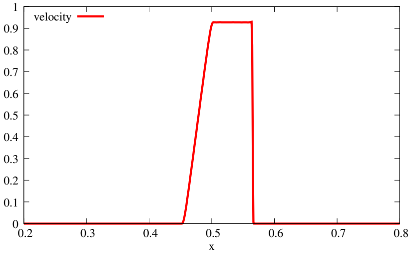

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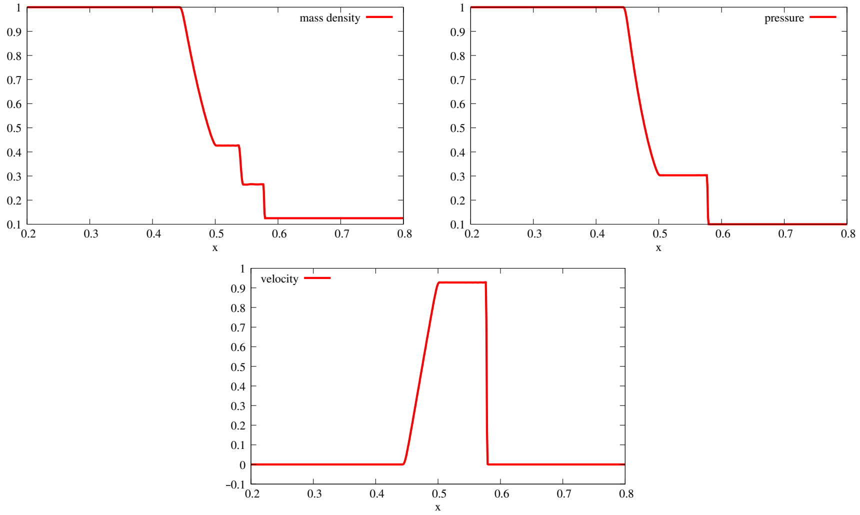

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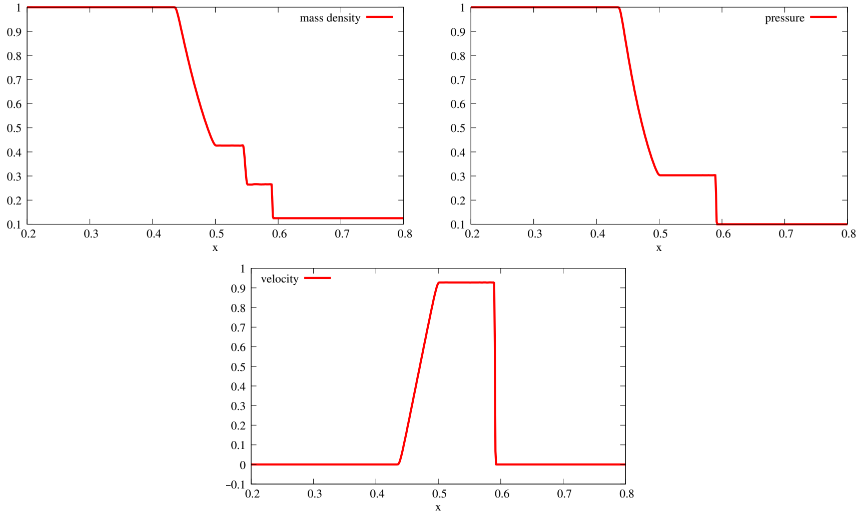

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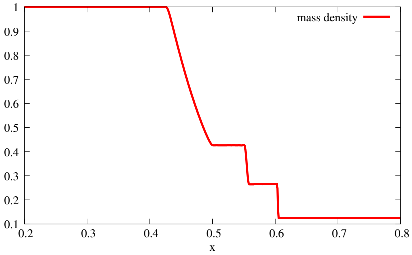

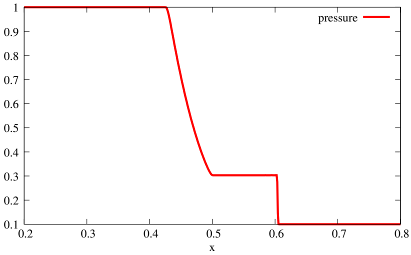

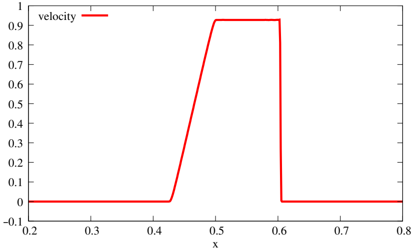

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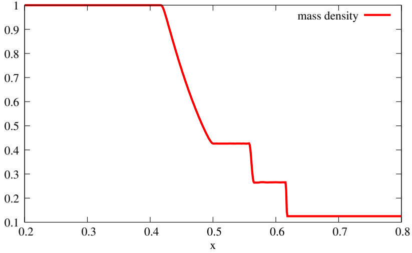

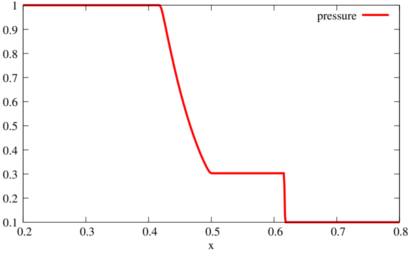

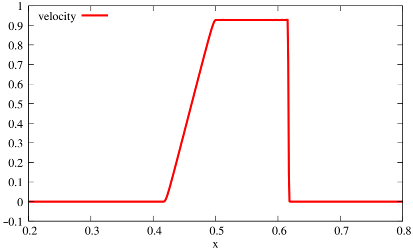

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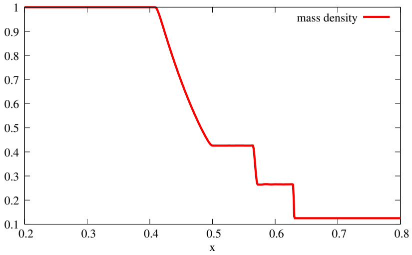
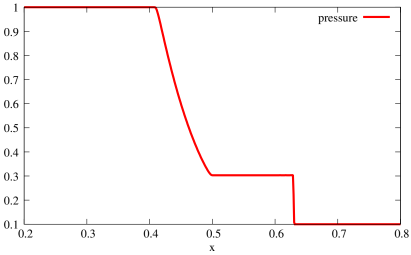
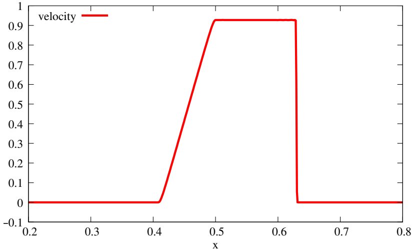
Example for one realisation of the interface position (deterministic)







A simple hydrodynamical example ('fil rouge 1')

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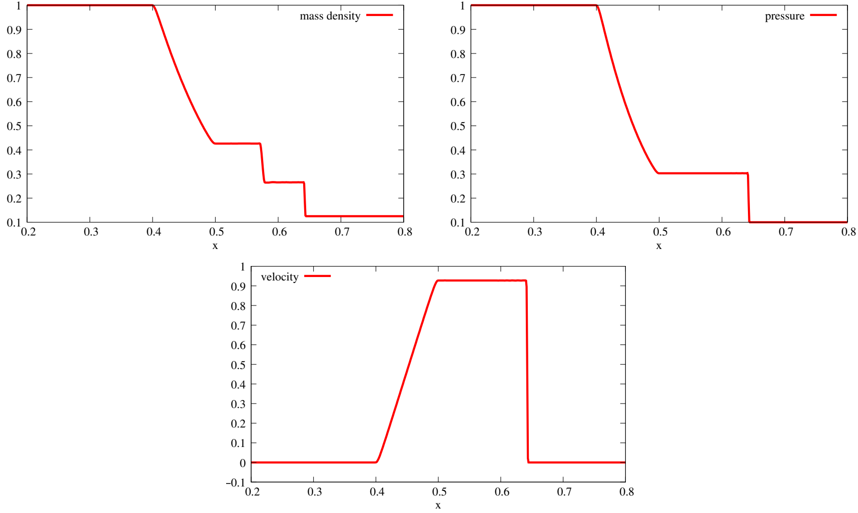

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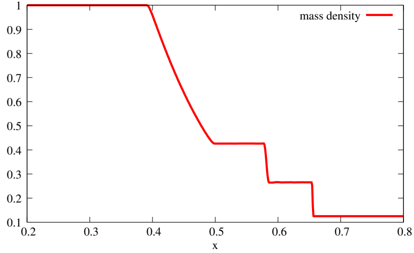

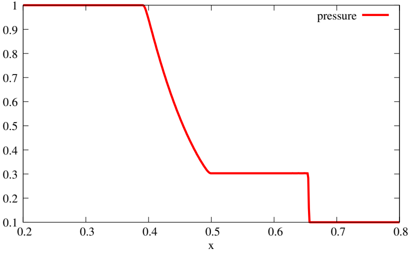

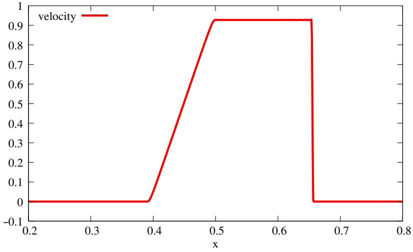

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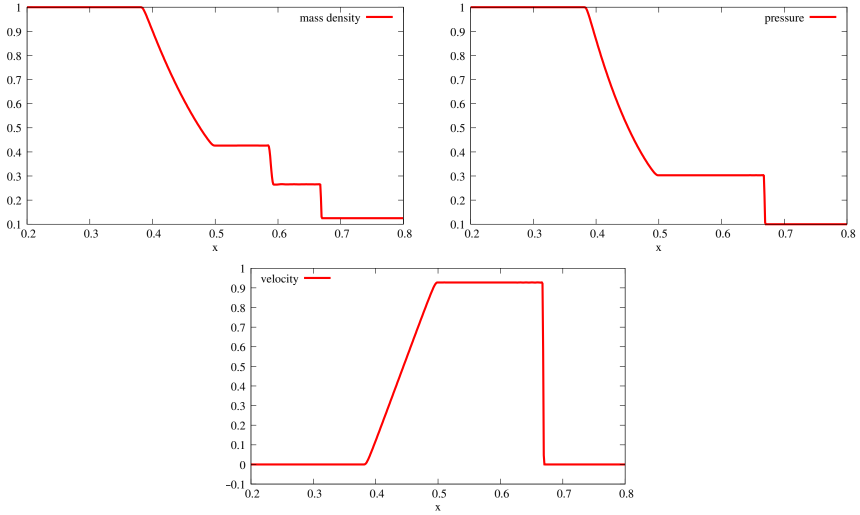
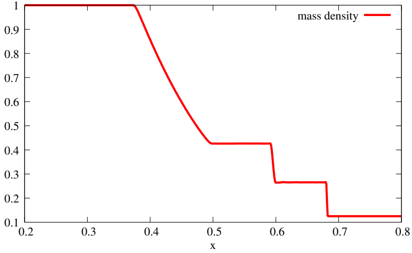
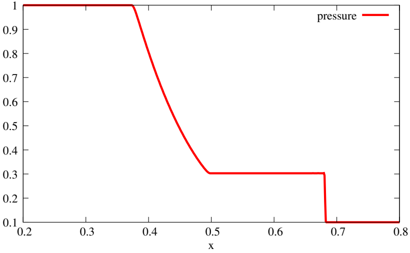
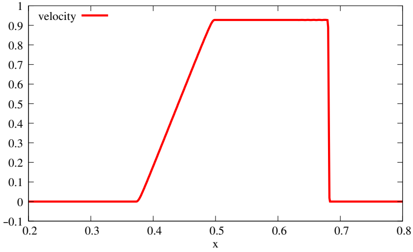




mass density
x
0.2 0.3 0.4 0.5 0.6 0.7 0.8
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

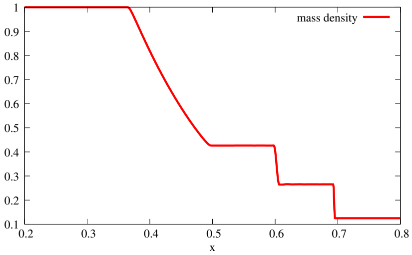


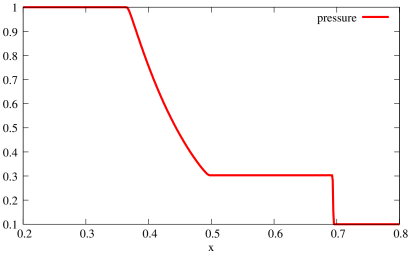
pressure
x
0.2 0.3 0.4 0.5 0.6 0.7 0.8
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

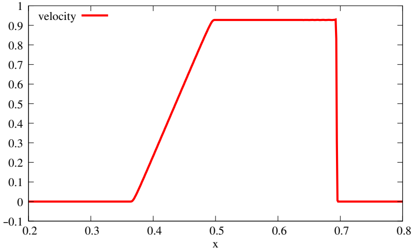


velocity
x
0.2 0.3 0.4 0.5 0.6 0.7 0.8
−0.1 0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

A simple hydrodynamical example ('fil rouge 1')
Example for one realisation of the interface position (deterministic)







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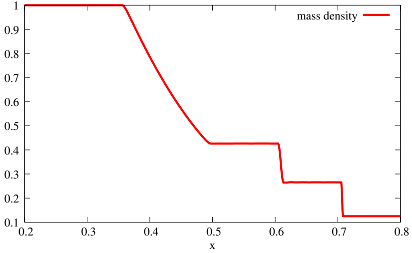
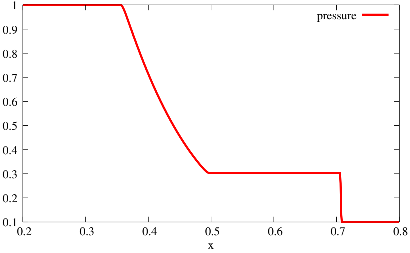
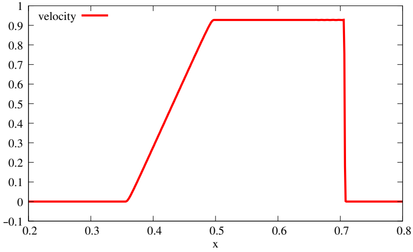
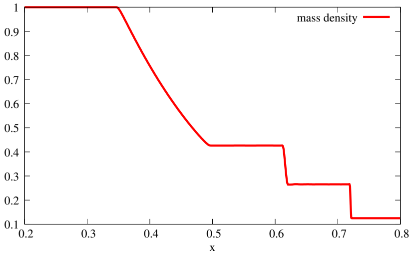
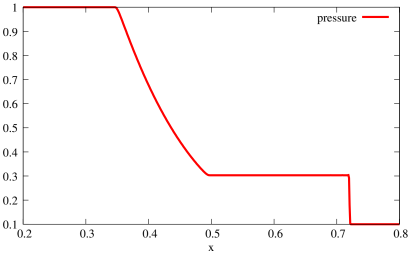




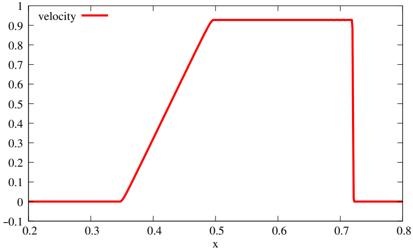




mass density
x



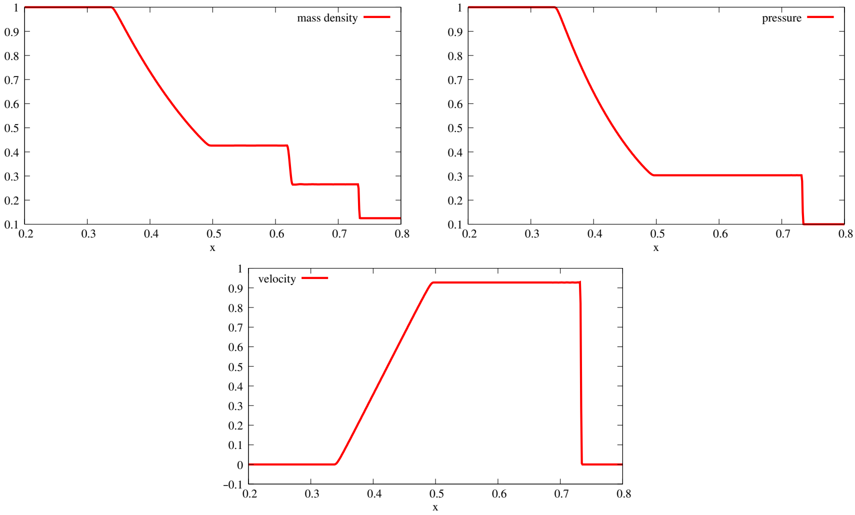
pressure
x



velocity
x

A simple hydrodynamical example ('fil rouge 1')

Example for one realisation of the interface position (deterministic)



A more relevant physical configuration ('fil rouge 2')

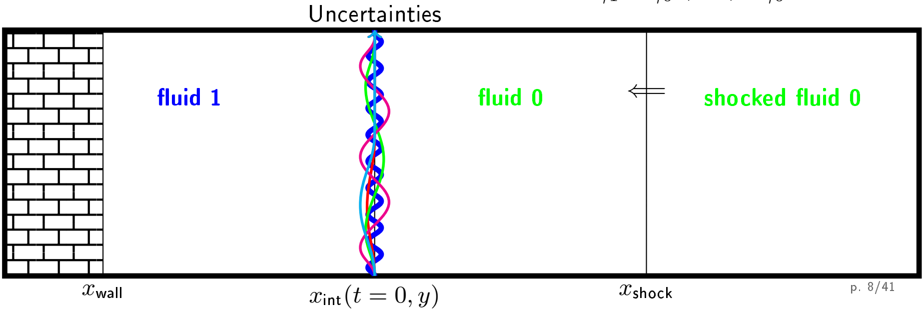
A (2D) Richtmyer-Meshkov shock tube

- Multi fluid Euler system (α volumic fraction):

$$\begin{cases} \partial_t \rho\alpha + \partial_x \rho v_x \alpha + \partial_y \rho v_y \alpha = 0, \\ \partial_t \rho + \partial_x \rho v_x + \partial_y \rho v_y = 0, \\ \partial_t \rho v_x + \partial_x(\rho v_x^2 + p) + \partial_y(\rho v_x v_y) = 0, \\ \partial_t \rho v_y + \partial_x(\rho v_x v_y) + \partial_y(\rho v_y^2 + p) = 0, \\ \partial_t \rho e + \partial_x(\rho v_x e + p v_x) + \partial_y(\rho v_y e + p v_y) = 0. \end{cases}$$

- Perfect gas closure, additivity of internal energy + isobaric at the interface:

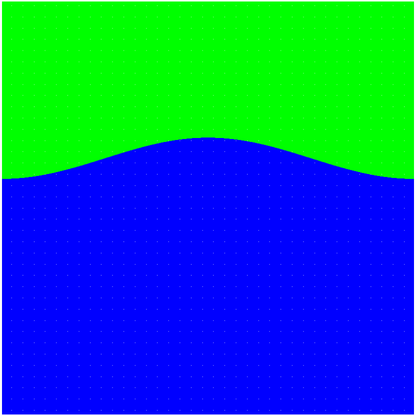
$$p(\rho, \varepsilon, \alpha) = (\Gamma(\alpha) - 1)\rho\varepsilon, \text{ with } \Gamma(\alpha) = -\frac{\gamma_0\gamma_1 - \gamma_1 + \alpha\gamma_1 - \alpha\gamma_0}{-\alpha\gamma_1 - \gamma_0 + 1 + \alpha\gamma_0}.$$



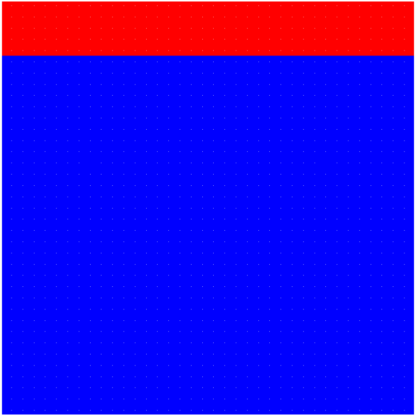
The Richtmyer-Meshkov instability ('fil rouge 2')

Example for one realisation of the interface position (deterministic)

Volume fraction $\alpha(x, y, t = 0ms)$

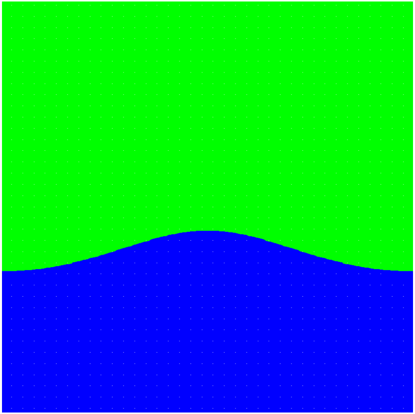


Pressure $p(x, y, t = 0ms)$

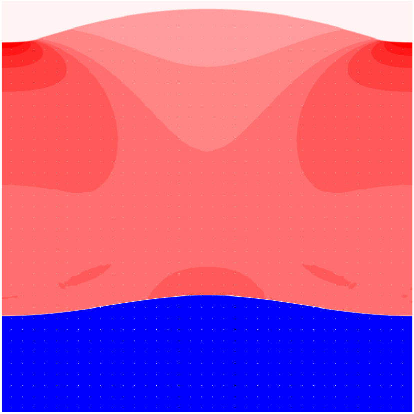


Example for one realisation of the interface position (deterministic)

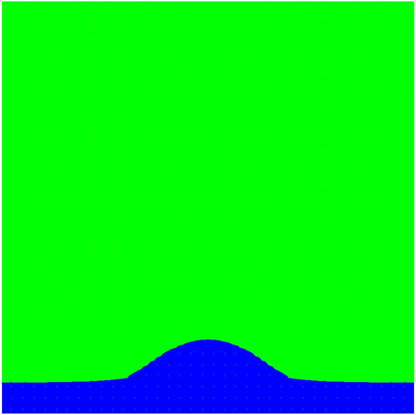
Volume fraction $\alpha(x, y, t = 2ms)$



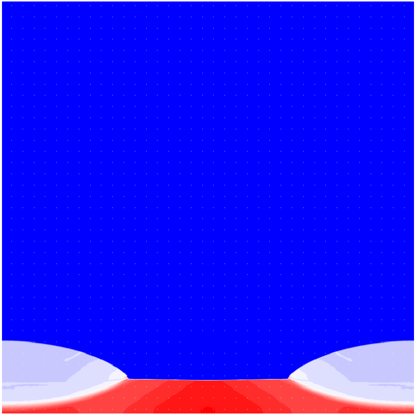
Pressure $p(x, y, t = 2ms)$



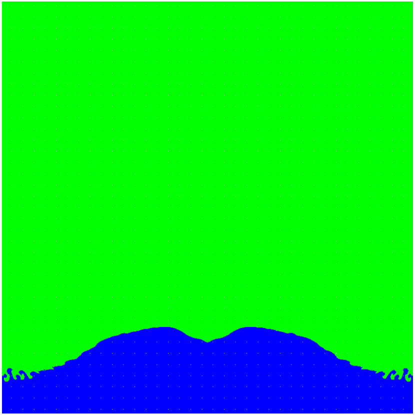
Volume fraction $\alpha(x, y, t = 4ms)$



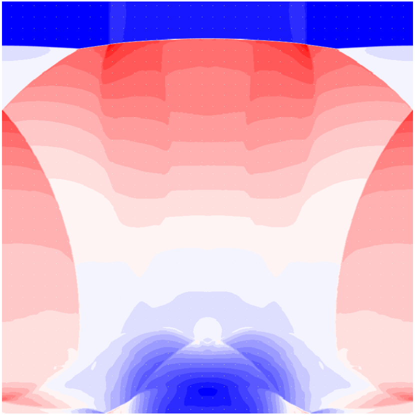
Pressure $p(x, y, t = 4ms)$



Volume fraction $\alpha(x, y, t = 6ms)$



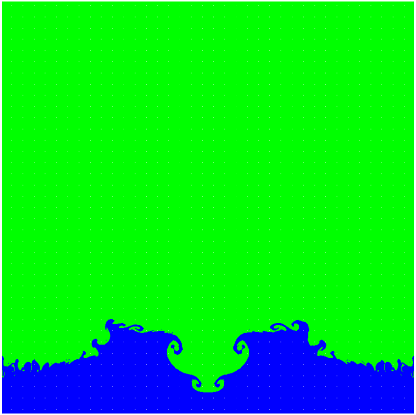
Pressure $p(x, y, t = 6ms)$



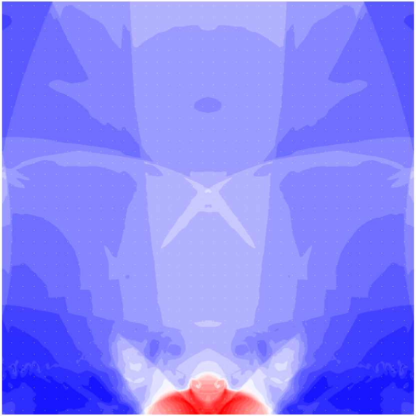
The Richtmyer-Meshkov instability ('fil rouge 2')

Example for one realisation of the interface position (deterministic)

Volume fraction $\alpha(x, y, t = 8ms)$

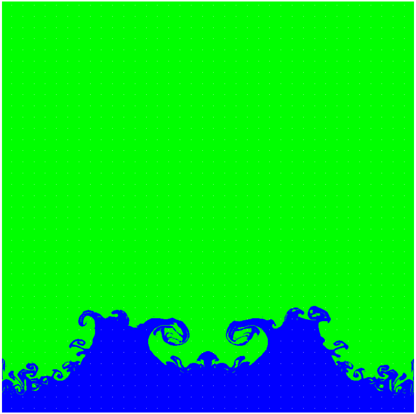


Pressure $p(x, y, t = 8ms)$

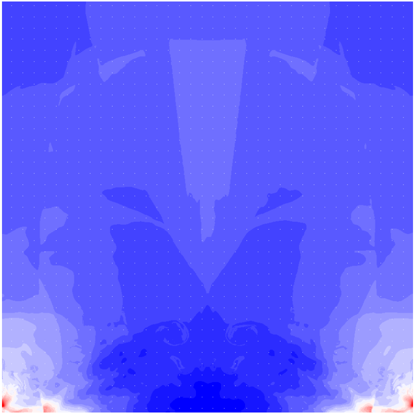


Example for one realisation of the interface position (deterministic)

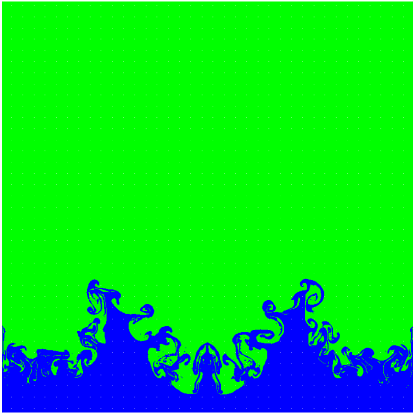
Volume fraction $\alpha(x, y, t = 10ms)$



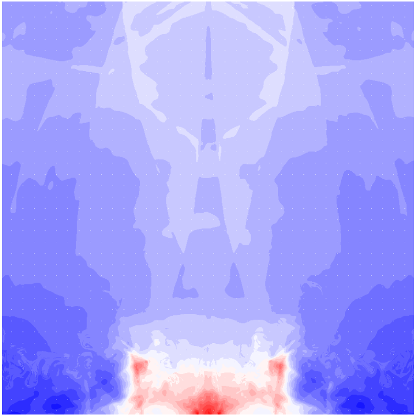
Pressure $p(x, y, t = 10ms)$



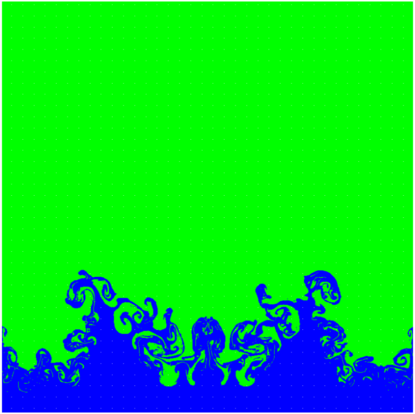
Volume fraction $\alpha(x, y, t = 12ms)$



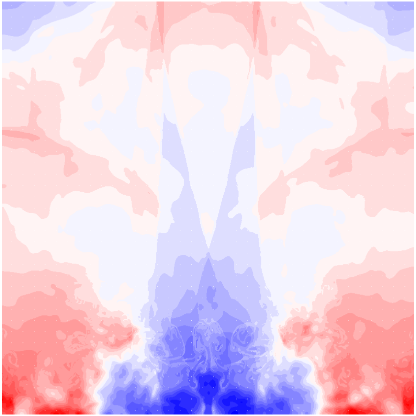
Pressure $p(x, y, t = 12ms)$



Volume fraction $\alpha(x, y, t = 14ms)$



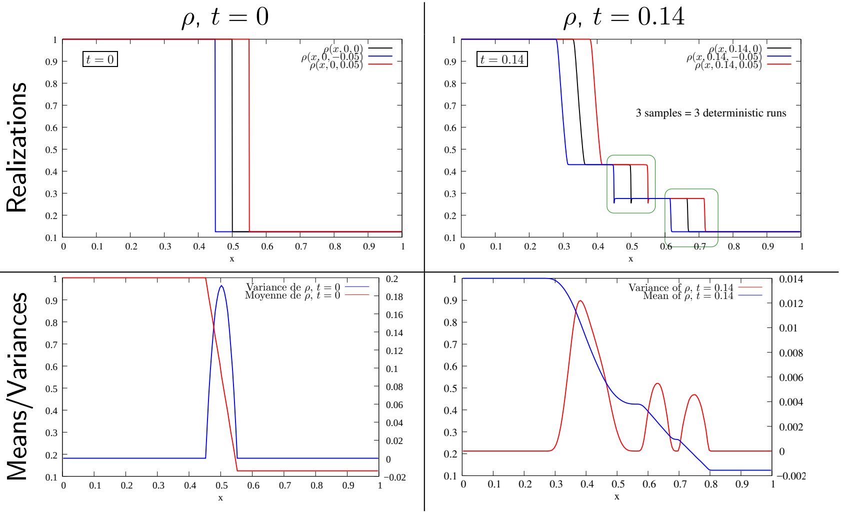
Pressure $p(x, y, t = 14ms)$



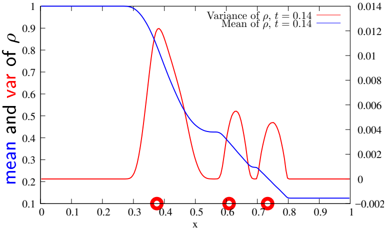
The 'fil rouge 1' uncertainty propagation problem

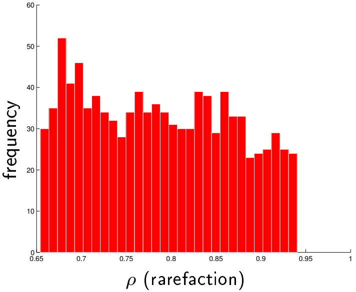
f_{post}: *mean and variance of the mass density* ρ *at* $t = 0.14$

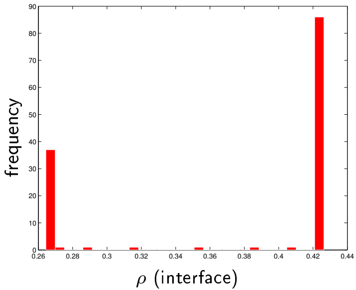
Initial uncertainty $x_{\text{int}}(t = 0, \Xi) \sim \mathcal{U}([0.45, 0.55])$, MC resolution $N_{MC} = 1000$

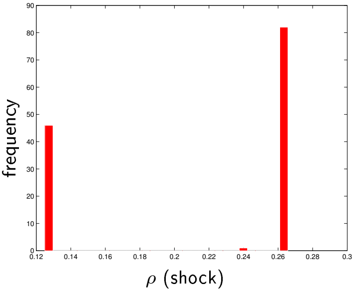


f_{post}: *pdfs of* ρ *for the different waves at* $t = 0.14$ *(Monte-Carlo)*









1. Introduction and Motivations
   - A deterministic configuration...
   - ... made uncertain
2. How to build uncertainty capturing reduced models
   - Burgers' equation and the fast convergence of the reduced models
   - Shallow water's equations and the loss of hyperbolicity
   - A new moment-based closure
   - Back to our 'fil rouge' problem: Euler's system
3. Conclusion

Reduced Models for uncertainty propagation
(those models are generally called intrusive*)*

- General methods to build a reduced model: define the set

$$L^2_\Xi(\Omega) = \left\{ \text{mesurable function } \Xi \to u(\Xi) \middle| \int u^2(\Xi)\mathrm{d}\mathcal{P}_\Xi < \infty \right\},$$

  and consider a basis of this space $\Xi \to \phi_k(\Xi), k \in \mathbb{N}$.
  We look for an approximation of this solution on the previous basis:

$$u^P(x,t,\Xi) = \sum_{k=0}^{P} u_k(x,t)\phi_k(\Xi), \text{ where } u_k(x,t) = \int u(x,t,\Xi)\phi_k(\Xi)\mathrm{d}\mathcal{P}_\Xi.$$

  Injecting u^P in the SCL of interest and taking its moments w.r.t. ϕ_k:

$$\partial_t u_k + \partial_x \int f\left(\sum_{k=0}^{P} u_k\phi_k\right)\mathrm{d}\mathcal{P}_\Xi = 0, \forall k \in \{0,...,P\}.$$

- Note that such methodology implies several constraints w.r.t. to the stochastic dimension and regularity of the solution (compared to MC).
- Very general methodology: depending on the choice of the basis, called Polynomial Chaos [7], Wavelet [12], ME-gPC [27, 6], WENO-based [1, 23]...

Let's have a step by step approach with increasing complexity:

- A scalar conservation law: Burgers' equation,
- Shallow water system,
- Euler system and general SCL.

We aim at presenting the construction of a reduced models for each conservation laws and study their properties.

Uncertain Burgers'equation and hyperbolicity

Let us consider an uncertain problem for Burgers' equation:

$$\left\{\begin{array}{l} \partial_t u(x,t,\Xi) + \partial_x \left(\dfrac{u^2(x,t,\Xi)}{2} \right) = 0, \\ u(x,t=0,\Xi) = u_0(x,\Xi), \Xi \sim \mathcal{U}([-1,1]) \end{array}\right.$$

Let us solve this problem with gPC as described in the litterature:

- We introduce the orthonormal basis $(\phi_k)_{k\in\mathbb{N}}$ with respect to d$\mathcal{P}_\Xi$.
- We consider the expansion $u^P = \sum_{k=0}^{P} u_k \phi_k$.
- We inject u^P in the previous system and perform a Galerkin projection on $(\phi_k)_{k\in\mathbb{N}}$ with respect to the measure of Ξ d$\mathcal{P}_\Xi$.
- We obtain:

$$\left\{\begin{array}{l} \partial_t u_0 + \partial_x \int_\Omega \dfrac{\left(\sum_{0\leq q\leq P} u_q\phi_q\right)^2}{2} \phi_0 \mathrm{d}\mathcal{P}_\Xi = 0, \\ \dots \\ \partial_t u_P + \partial_x \int_\Omega \dfrac{\left(\sum_{0\leq q\leq P} u_q\phi_q\right)^2}{2} \phi_P \mathrm{d}\mathcal{P}_\Xi = 0. \end{array}\right.$$

The Jacobian of the the flux A is of size $P \times P$ with $A_{i,j} = \displaystyle\int u^P \phi_i \phi_j \mathrm{d}\mathcal{P}_\Xi$: it is symmetric$\Longrightarrow$ hyperbolic reduced model.

Uncertain Burgers'equation and proof of spectral accuracy

Let us consider an uncertain problem for Burgers' equation:

$$\begin{cases} \partial_t u(x,t,\Xi) + \partial_x \left(\dfrac{u^2(x,t,\Xi)}{2} \right) = 0, \\ u(x,t=0,\Xi) = u_0(x,\Xi), \Xi \sim \mathcal{U}([-1,1]) \\ \text{on the periodic domain } x \in [0,1]_{\text{per}}. \end{cases}$$

The initial data is supposed to be smooth function for all Ξ and we assume the time

$$T_\Xi = -\frac{1}{\inf\limits_x \left(\partial_x u_0(x,\Xi)\right)}.$$

at which a discontinuous solution appears is bounded from below uniformly

$$\exists T, \quad 0 < T < T_\Xi \ \ \forall \Xi.$$

We also assume the exact solution is smooth with respect to all variables

$$u \in L^\infty\left((0,1)\times(0,T^\varepsilon)\times(-1,1)\right) \cap L^\infty\left([0,1]_{\text{per}} \times (0,T^\varepsilon) : H^k(-1,1)\right)$$

for all $k \in \mathbb{N}$ where

$$H^k(\Omega) = \left\{ u \in L^2(\Omega) | \int \sum_{l=0}^{k} (u^{(l)})^2 d\mathcal{P}_\Xi < \infty \right\}.$$

Theorem (Convergence of Burgers' approximation)

Spectral accuracy holds in the following sense: if we denote by

$$\|u(t)\|^2_{L^2(\mathcal{I}\times\Omega)} = \int_{\mathcal{I}}\int_{\Omega} u^2(x,t,\xi)\mathrm{d}\mathcal{P}_{\Xi}(\xi)\mathrm{d}x$$

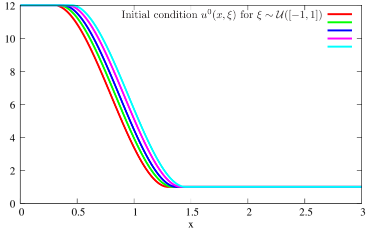
for all k there exists a constant D_k^ε such that

$$\left\|u(t) - u^P(t)\right\|^2_{L^2(\mathcal{I}\times\Omega)} \leq D_k^\varepsilon\left(\left\|u(0) - u^P(0)\right\|^2_{L^2(\mathcal{I}\times\Omega)} + \frac{1}{P^k}\right), \qquad t \leq T^\varepsilon.$$

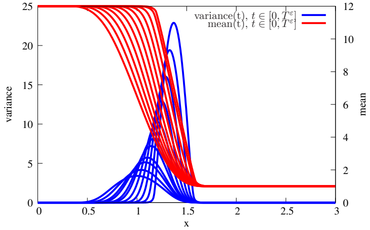
Références : [4]

Uncertain Burgers'equation and proof of spectral accuracy

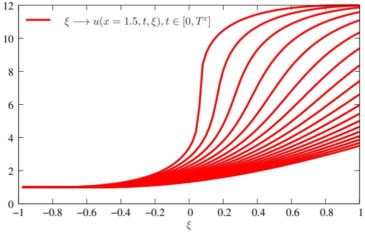
$t = 0$



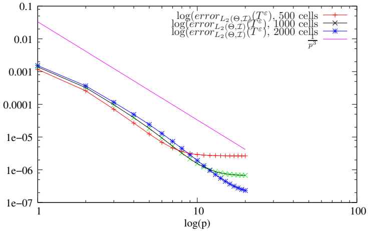
$t = T$



time evolution in uncertain space



Spectral convergence



More details on this test-case in Després *et al.* (2013)

Shallow water system

$$\begin{cases} \partial_t h + \partial_x (hv) = 0, \\ \partial_t (hv) + \partial_x \left(hv^2 + g\frac{h^2}{2} \right) = 0, \end{cases}$$

where $h(x,t,\Xi)$ is the water height, $v(x,t,\Xi)$ is the velocity of the water column and $g > 0$ is the gravity constant.

The jacobian of the flux is not symmetric $\Longrightarrow$ hyperbolicity is not straightforward.

Theorem (Non hyperbolicity of the Shallow water truncated system)

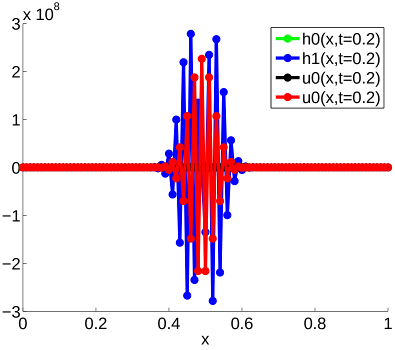
The truncated system obtained from the Shallow water system together with the previous Galerkin projection is not always hyperbolic.

For $P = 1$ as a truncature order, the physical state $u_0 = (1,0)$ and $u_1 = \left(0, \sqrt{\frac{1}{3}}\right)$ with $0 < g < \frac{3}{25}$ leads to non physical solutions.

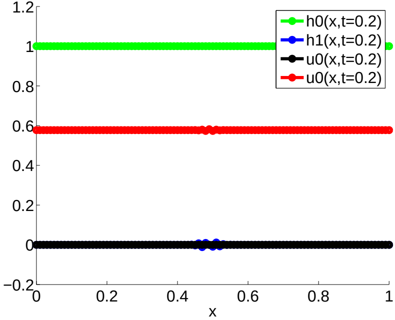
Shallow water problem

- What happens when one tries to numerically solve a non hyperbolic problem?
- initialization of the stationnary state $u_0 = (1,0)$ and $u_1 = \left(0, \sqrt{\frac{1}{3}}\right)$.
- Numerical resolution for 2 values of the numerical diffusion coefficient:

Reference $D_n = 1$



$D_n = 100000$



$\Longrightarrow$ Numerical diffusion can artificially control the oscillations and makes the solution look physical whereas it is not.

The closure problem

- Is it possible to close the truncated system so that the hyperbolicity of the reduced model is ensured?
- An element of answer: analogy with kinetic theory and moment models.

Références : [15]

Kinetic Theory	$\longrightarrow$	Uncertainty Propagation
$\partial_t h + v\partial_x h = C(h,h)$	$\longrightarrow$	$\partial_t u(x,t,\Xi) + \partial_x f(u(x,t,\Xi)) = 0$
$h(x,t,v)\mathrm{d}v$	$\longrightarrow$	$u(x,t,\Xi)\mathrm{d}\mathcal{P}_\Xi$
Multiplicators $\Phi(v) = 1, v, v^2, \ldots$	$\longrightarrow$	Multiplicators $\Phi(\Xi) = \phi_0(\Xi), \phi_1(\Xi), \ldots$

Moments of h:

$$\begin{aligned} \rho &= \int h \ \mathrm{d}v &= \langle 1 \rangle \\ \rho u &= \int h \ v \ \mathrm{d}v &= \langle v \rangle \\ &\cdots \end{aligned}$$

$\longrightarrow$

Moments of u:

$$\begin{aligned} u_0 &= \int u\phi_0 \mathrm{d}\mathcal{P}_\Xi \\ u_1 &= \int u\phi_1 \mathrm{d}\mathcal{P}_\Xi \\ &\cdots \end{aligned}$$

reduced model for h

$$\begin{cases} \partial_t \langle 1 \rangle + \partial_x \langle v \rangle &= 0, \\ \ldots & \\ \partial_t \left\langle v^k \right\rangle + \partial_x \left\langle v^{k+1} \right\rangle &= 0, \\ \ldots & \\ \partial_t \left\langle v^P \right\rangle + \partial_x \left\langle v^{P+1} \right\rangle &= 0. \end{cases}$$

$\longrightarrow$

Reduced model for u

$$\begin{cases} \partial_t u_0 + \partial_x f_0 = 0, \\ \ldots \\ \partial_t u_k + \partial_x f_k = 0, \\ \ldots \\ \partial_t u_P + \partial_x f_P = 0. \end{cases}$$

$\Longrightarrow$ The systems are not closed yet.

Closure: analogy with Moment theory

- Previous result: P_N closure are not enough to preserve hyperbolicity (see Shallow water)
- Moment Theory : resolution of a underdeterminanted inverse problem

$$\text{Find } u \in L^2(\Omega, \mathcal{F}, \mathcal{P}) \ / \left\{ \begin{array}{c} \displaystyle\int u\ \phi_0\ \mathrm{d}\mathcal{P}_\Xi = u_0, \\ \dots, \\ \displaystyle\int u\ \phi_k\ \mathrm{d}\mathcal{P}_\Xi = u_k, \\ \dots, \\ \displaystyle\int u\ \phi_P\ \mathrm{d}\mathcal{P}_\Xi = u_P, \end{array} \right.$$

where $(\phi_i)_{i\in\{0,\dots,P\}}$ are real functions on Ω, basis of $L^2(\Omega, \mathcal{F}, \mathcal{P})$.

- in this presentation: $(\phi_i)_{i\in\{0,\dots,P\}}$ = gPC basis.
- Non unicity of the distribution u

$\Longrightarrow$ In kinetic theory: closure entropy (Shannon): $\eta(u) = \displaystyle\int u \ln(u) \mathrm{d}\mathcal{P}_\Xi$.

- Find u as the minimum of η under the constraints $\Longrightarrow$ Unicity of u.
  Find the Lagrange multiplicators $(\lambda_k)_{k\in\{0,\dots,P\}}$ minimizing

$$T(\lambda_0, .., \lambda_P) = -\int \eta(u(\lambda_0, .., \lambda_P))\mathrm{d}\mathcal{P}_\Xi + \sum_{k=0}^{P} \int u(\lambda_0, .., \lambda_P)\lambda_k\phi_k \mathrm{d}\mathcal{P}_\Xi - \sum_{k=0}^{P} u_k\lambda_k.$$

Références : [13, 8]

Intrusive method based on moment closure

- The new obtained reduced problem:

$$\begin{cases} \partial_t u_0(\lambda_0,..,\lambda_P) + \partial_x f_0(\lambda_0,..,\lambda_P) = 0, \\ \dots \\ \partial_t u_k(\lambda_0,..,\lambda_P) + \partial_x f_k(\lambda_0,..,\lambda_P) = 0, \\ \dots \\ \partial_t u_P(\lambda_0,..,\lambda_P) + \partial_x f_P(\lambda_0,..,\lambda_P) = 0. \end{cases}$$

where $(\lambda_k)_{k\in\{0,\dots,P\}}$ minimize (closure)

$$T(\lambda_0,..,\lambda_P) = -\int \eta(u^P(\lambda_0,..,\lambda_P))\mathrm{d}\mathcal{P}_\Xi + \sum_{k=0}^{P}\int u^P(\lambda_0,..,\lambda_P)\lambda_k\phi_k\mathrm{d}\mathcal{P}_\Xi - \sum_{k=0}^{P} u_k\lambda_k.$$

- $\Longrightarrow$ Functional variation with respect to u^P leads to :

$\boxed{\nabla_u\eta(u^P(\lambda_0,..,\lambda_P)) = \sum_{k=0}^{P}\lambda_k\phi_k,}$ i.e. $u^P(\lambda_0,..,\lambda_P) = (\nabla_u\eta)^{-1}\left(\sum_{k=0}^{P}\lambda_k\phi_k\right).$

- <u>Back to our reduced model</u>: this is valid for every strictly convex entropy η.
  - $\eta(u) = \frac{u^2}{2}$, then $\nabla_u\eta(u^P) = u = \sum_k \lambda_k\phi_k \Longrightarrow$ we recover the classical approach.
  - $\eta(\tau) = \tau\ln(\tau) - \tau$, then $(\nabla_\tau\eta)^{-1}(\sum_k \lambda_k\phi_k) = \tau^P(\sum_k\lambda_k\phi_k) = e^{\sum_k\lambda_k\phi_k} > 0.$
  - $\eta = s$, i.e. closure entropy $=$ mathematical entropy,
    then $\nabla_u s(u^P) = \sum_k v_k\phi_k \approx v =$ is the entropic variable.

Références : [14]

Hyperbolicity of the moment based reduced model

Theorem: If $\eta = s$ then

- The reduced model has a strictly convex entropy:
$$(S^P, G^P) = \left(\int s(u^P) \mathrm{d}\mathcal{P}_\Xi, \int g(u^P) \mathrm{d}\mathcal{P}_\Xi \right).$$
- The system is hyperbolic $\forall P \in \mathbb{N}$.

Références : [17]

Brief summary:

- We perform a variable change which symmetrizes our system.
- Ensures the respect of mathematical/physical properties of the system.
- Needs a new step for the resolution.

Description of the resolution algorithm

<table>
<tr><td>beginning of time step t^n</td><td></td></tr>
<tr><td>the moments $u^n_{k,i}$ and the fluxes $f^*_{k,i}$ are known in each cells i.</td><td></td></tr>
<tr><td>Resolution of a high order system: $u^n_{k,i} \to u^{n+1}_{k,i}$</td><td rowspan="2">Common

with gPC</td></tr>
<tr><td>$$\frac{1}{\Delta t}\begin{pmatrix} u^{n+1}_{0,j} - u^n_{0,j} \\ \cdots \\ u^{n+1}_{P,j} - u^n_{P,j} \end{pmatrix} + \frac{1}{\Delta x}\begin{pmatrix} f^*_{0,d} - f^*_{0,g} \\ \cdots \\ f^*_{P,d} - f^*_{P,g} \end{pmatrix} = 0.$$</td></tr>
<tr><td>Computation of $\Lambda^{n+1}_i = (\lambda^{n+1}_{0,i}, \cdots, \lambda^{n+1}_{P,i})^t$ from $U^{n+1}_i = (u^{n+1}_{0,i}, \cdots, u^{n+1}_{P,i})$?</td><td rowspan="5">Specific

to the moment based method</td></tr>
<tr><td>Minimization of a str. convex functional $T(\Lambda) = -\left\langle U^{n+1}_i, \Lambda \right\rangle + \langle U(\Lambda), \Lambda\rangle - S(U(\Lambda))$.
Newton (quadratic convergence) $\begin{cases} \text{- } \Lambda^k \to \Lambda^{k+1}, \\ \text{- } \Lambda^{k+1} = \Lambda^k - T'^{-1}(\Lambda^k)T(\Lambda^k), \\ \text{- } \vert\vert\Lambda^{k+1} - \Lambda^k\vert\vert < \epsilon_{Newt} = 10^{-13}, \\ \text{- with } \Lambda^n_i \text{ as "guess".} \end{cases}$</td></tr>
<tr><td>Evaluation of the fluxes at the next time step</td></tr>
<tr><td>$$f^*_{k,j+1/2}(\ldots, \lambda^{n+1}_{0,i}, \ldots, \lambda^{n+1}_{P,i}, \lambda^{n+1}_{0,i+1}, \ldots, \lambda^{n+1}_{P,i+1}, \ldots)$$</td></tr>
<tr><td>end of time step t^{n+1}</td></tr>
</table>

$\Longrightarrow$ More details in Poëtte *et al.* (2009)

Références : [17]

Back to our hydrodynamical problem

- Compressible gas dynamics, 1D cartesian coordinates:

$$\left\{\begin{array}{l} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ \partial_t(\rho e) + \partial_x(\rho u e + p u) = 0, \end{array}\right.$$

  with a perfect gas closure: $p = (\gamma - 1)\rho\epsilon$.

- closure/mathematical entropy: $s(\rho, \rho u, \rho e) = -\rho \ln\left(\rho^{-\gamma}(\rho e - \frac{(\rho u)^2}{2\rho})\right)$.

  $\Longrightarrow$ The reduced system is hyperbolic $\diamond$

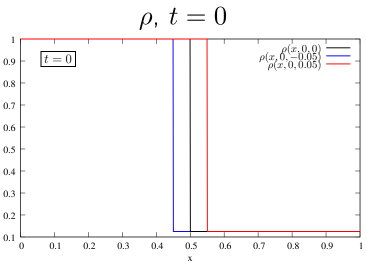
- Expression of $(\rho, \rho u, \rho e)^t$ with respect to the entropic variable $V = (v_1, v_2, v_4)^t$.

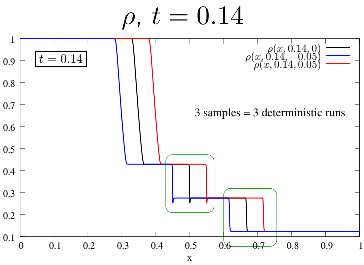
$$\begin{pmatrix} \rho(V) \\ \rho u(V) \\ \rho e(V) \end{pmatrix} = \begin{pmatrix} e^{\frac{2\,v_1\,v_4 - 2\,v_4\,\ln(-v_4) - 2\,v_4\,\gamma - {v_2}^2}{2 v_4 (\gamma - 1)}} \\ -\frac{v_2}{v_4} e^{\frac{2\,v_1\,v_4 - 2\,v_4\,\ln(-v_4) - 2\,v_4\,\gamma - {v_2}^2}{2 v_4 (\gamma - 1)}} \\ \frac{v_2^2 - 2v_4}{2v_4^2} e^{\frac{2\,v_1\,v_4 - 2\,v_4\,\ln(-v_4) - 2\,v_4\,\gamma - {v_2}^2}{2 v_4 (\gamma - 1)}} \end{pmatrix}.$$

$\Longrightarrow$ Positivity of the mass density ρ ensured even when V is polynomial $\diamond$

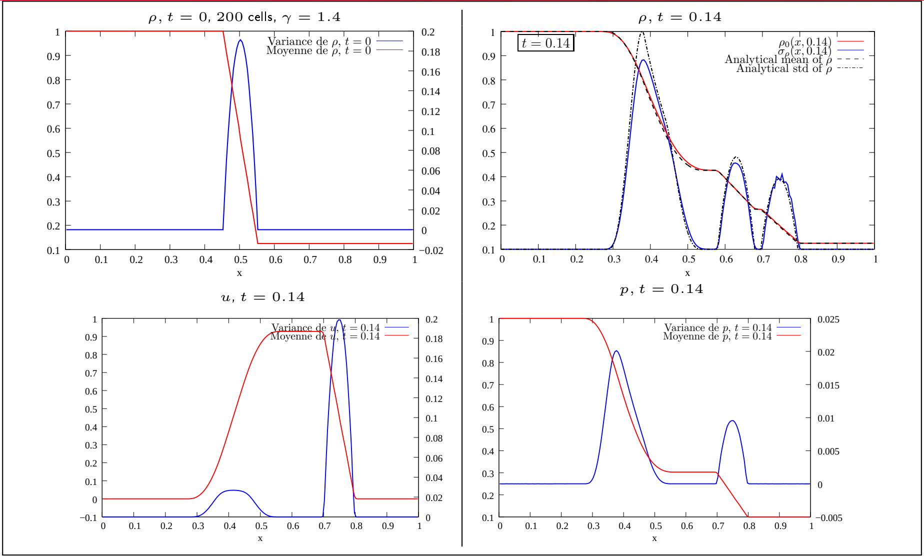
- Uncertain Sod shock tube (stochastic Riemann problem).
- Uncertain initial position of the interface: $\Xi \sim \mathcal{U}([0.45, 0.5])$.
- As time evolves three waves are propagating: a rarefaction fan, the interface and a shock.

Realizations





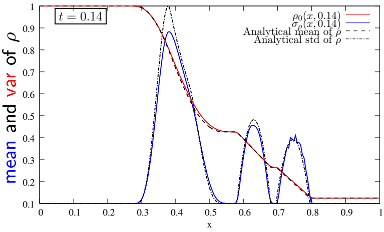
Euler 1-D: Stochastic Rieman Problem (uncertain interface)

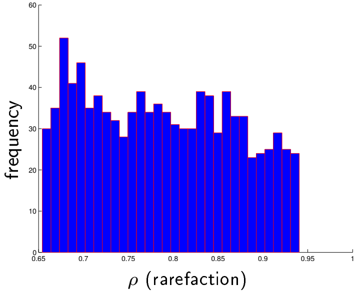


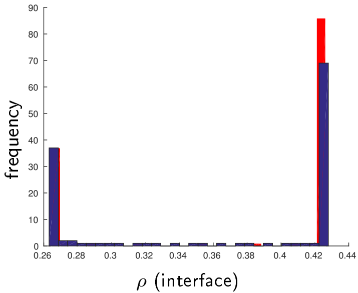
- Rarefaction fan, interface and shock "shares" the uncertain as time evolves.
- Velocity and pressure are continuous in the vicinity of the interface.

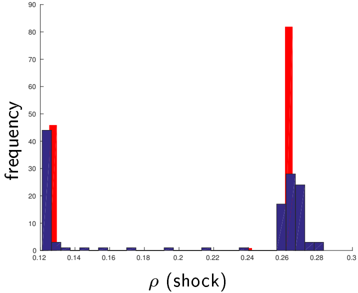
The 'fil rouge 1' uncertainty propagation problem

Comparisons MC vs. Moment-based reduced model $P = 20$

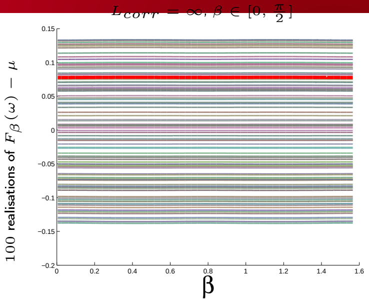






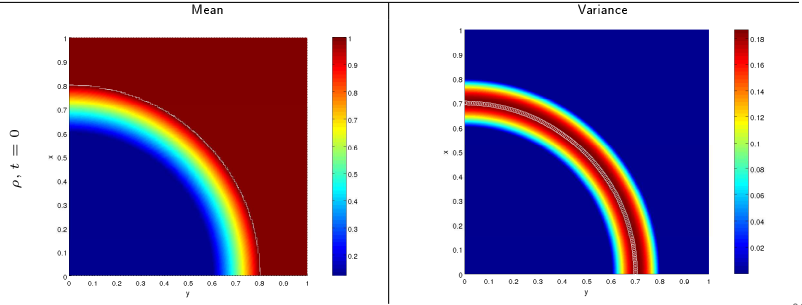


In axisymmetric geometry: $L_{corr} = \infty$, initial conditions



Particularities $L_{corr} = \infty$, $Q = 1$, $P = 20$, $N_q = 33$ points CC, 21 Moments.
Common characteristics:
3^{rd} order GAIA scheme,
with directional splitting.
Mesh 200×200,
$\gamma = 1.4$, $\mu = 0.7$, $\sigma = 0.08$,

BC: wall	$t = 0.185$ (Sod),
and neutral	$t = 0.74$ (RM).

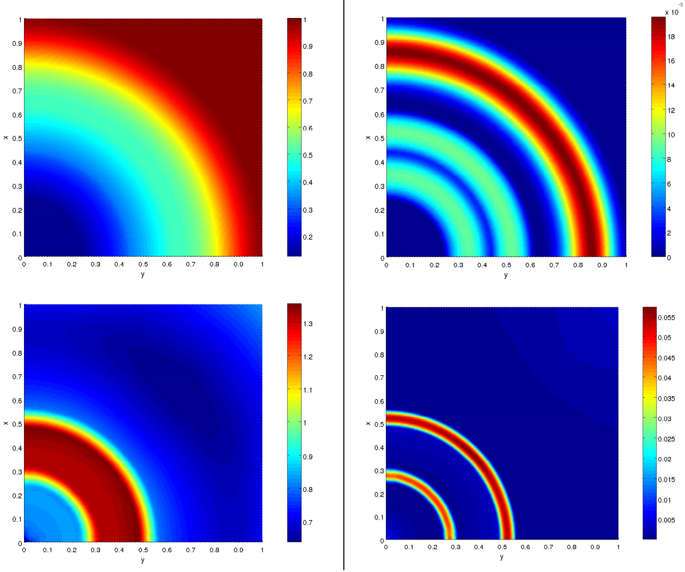


In axisymmetric geometry: $L_{corr} = \infty$, 200×200 cells

Moyenne | Variance

$\rho, t = 0.185$

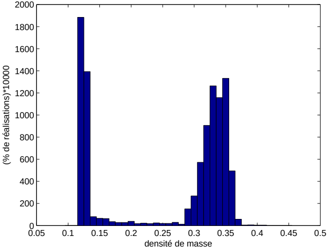
$\rho, t = 0.74$



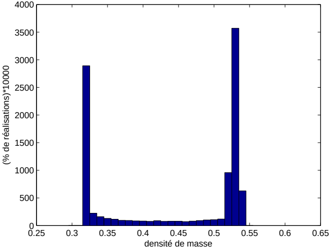
Pdfs of the mass density for the different waves $t = 0.14$

pdf of ρ $(P = 20)$

shock

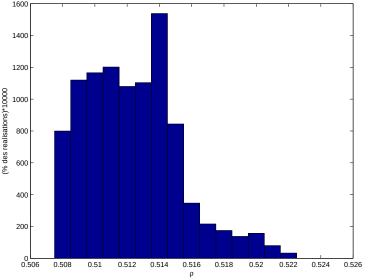


interface

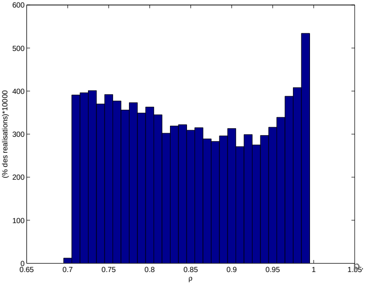


pdf of ρ $(P = 20)$

foot of the rarefaction



top of the rarefaction



- Construction of a moment closed reduced model for uncertainty propagation.
- Scalar conservation laws:
  - hyperbolicity is straightforward
  - proof of spectral convergence with respect to the truncature order P.
- Systems: possibility to build well-posed systems.
- Constructions remind of P_N and M_N models used in kinetic theory.
- Deepened study of the well-posed reduced models in Després *et al.* (2013) (wave velocity study, lagrangian case etc.)

Interesting recent related topics

- Spectral convergence for other PDEs? See for example [16].
- Hyperbolicity when a mathematical entropy is not available? See [4, 19, 10].
- Other ways to enforce hyperbolicity? Elements of answers in [5, 22, 20]
- Ways to enforce other properties (maximum principle)? See [9, 3].
- Is intrusiveness worth it? Elements of answers in [11, 2, 18, 21, 20].

Thanks for your attention !

Questions ?

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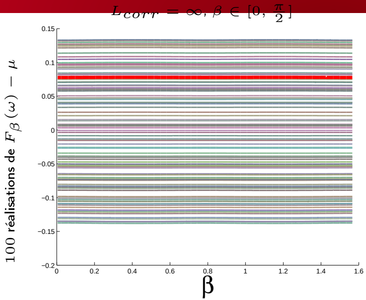
- Résultats Euler 2D (intrusif)

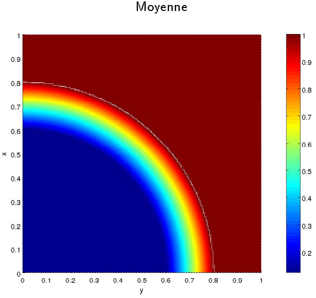
Solution de référence: $L_{corr} = \infty$, conditions initiales

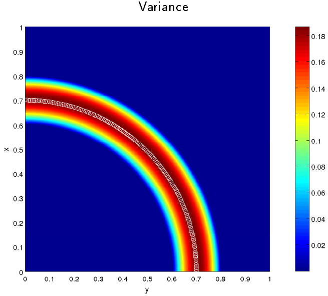


Particularités $L_{corr} = \infty$, $Q = 1$, $P = 20$, $N_q = 33$ points CC, 21 Moments.
Caractéristiques communes:
Schéma GAIA d'ordre 3,
avec Splitting directionnel.
Grille 200×200,
$\gamma = 1.4$, $\mu = 0.7$, $\sigma = 0.08$,

CB: Murs et Neutres	$t = 0.185$ (Sod), $t = 0.74$ (RM).

$\rho, t = 0$



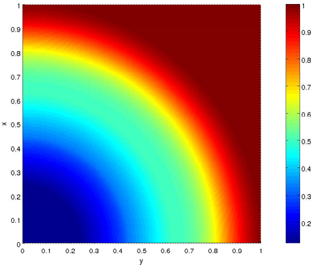


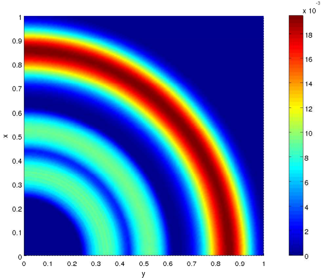
Solution de référence: $L_{corr} = \infty$, 200×200 mailles

Moyenne

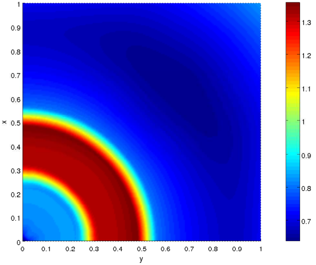
Variance

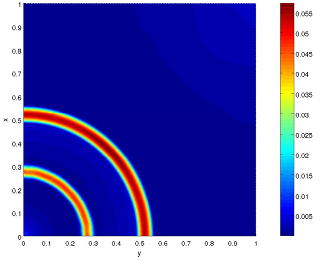
$\rho, t = 0.185$





$\rho, t = 0.74$



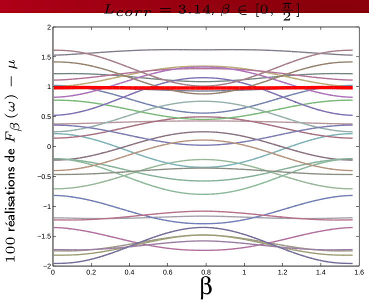


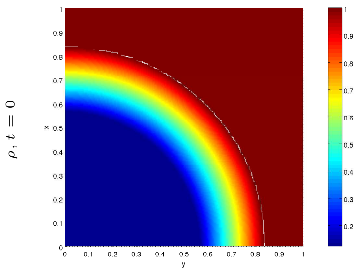
Interface perturbée, $L_{corr} = 3.14$, conditions initiales

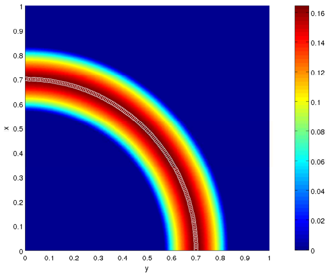
Particularités: $L_{corr} = 3.14$, $Q = 2$, $P = 3$, $N_q = 4^2 = 16$ points Gauss-Leg, 15 moments.

Caractéristiques communes Schéma GAIA d'ordre 3, avec Splitting directionnel.
Grille 200×200,
$\gamma = 1.4$, $\mu = 0.7$, $\sigma = 0.08$,

CB: Murs et Neutres	$t = 0.185$ (Sod), $t = 0.74$ (RM).





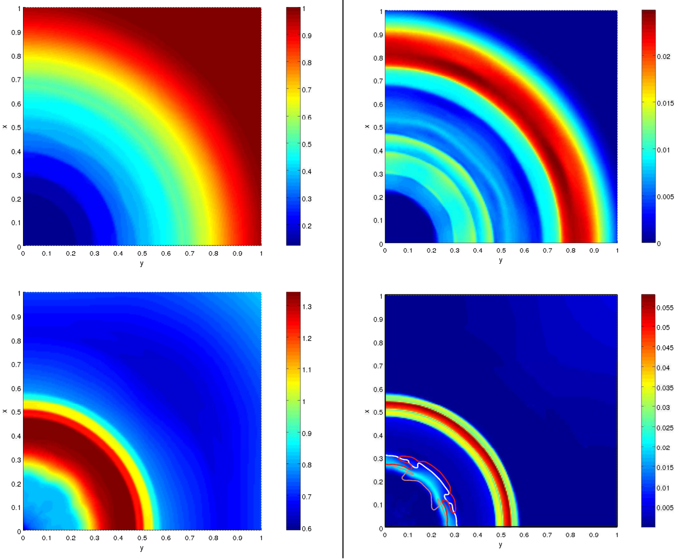


Interface perturbée, $L_{corr} = 3.14$, $Q = 3$

Moyenne | Variance

$\rho, t = 0.185$

$\rho, t = 0.74$

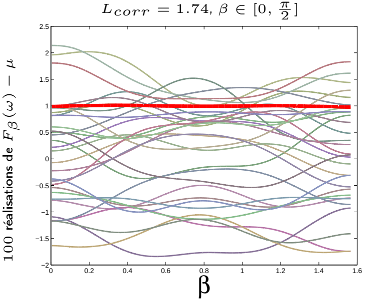


Interface perturbée, $L_{corr} = 1.74$, conditions initiales

Particularités: $L_{corr} = 1.74$, $Q = 5$, $P = 2$, $N_q = 3^5 = 243$, 21 moments.

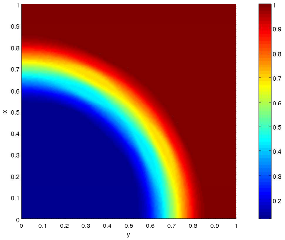
Caractéristiques communes | Schéma

GAIA d'ordre 3,
avec Splitting directionnel.
Grille 200×200,
$\gamma = 1.4$, $\mu = 0.7$, $\sigma = 0.08$.

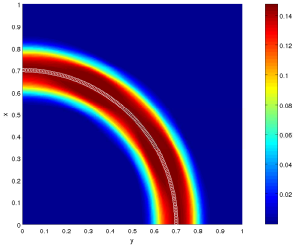


Moyenne

ρ, $t = 0$



Variance

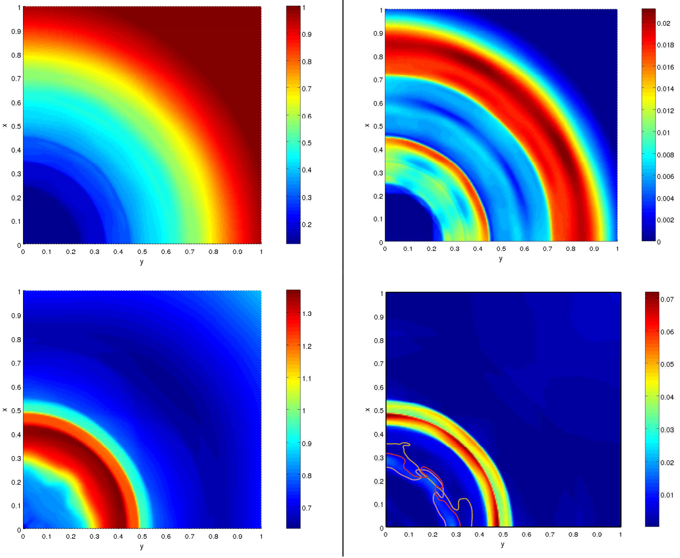


Interface perturbée, $L_{corr} = 1.74$, $Q = 5$

Moyenne | Variance

$\rho, t = 0.185$

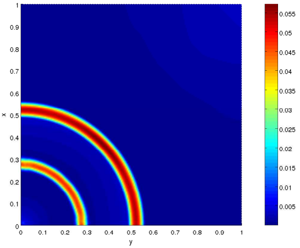
$\rho, t = 0.74$



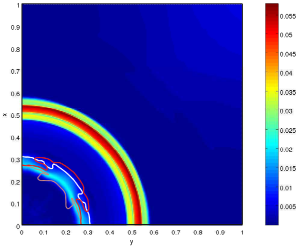
Variance de ρ et réalisations de la variable aléatoire 'interface' pour $\neq L_{corr}$

variance de ρ

$L_{corr} = \infty$

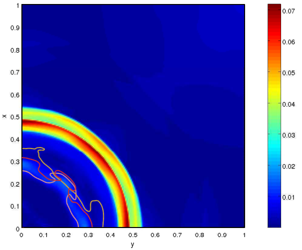


$L_{corr} = 3.14$

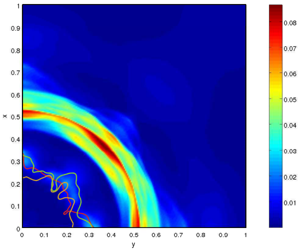


variance de ρ

$L_{corr} = 1.74$

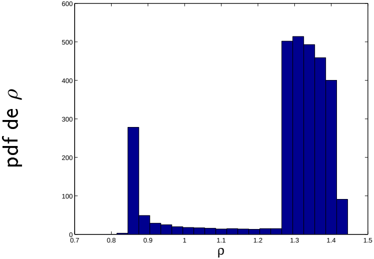


$L_{corr} = 0.31$

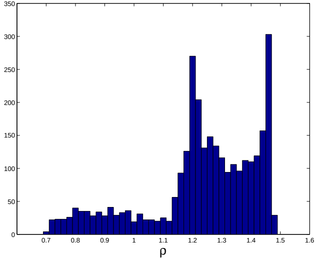


Pdfs de la densité de masse sur l'axe $x = y$ pour $\neq L_{corr}$

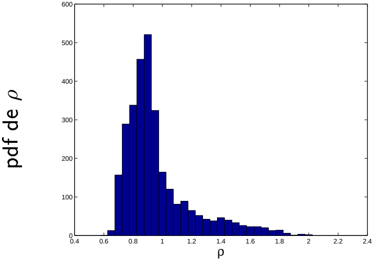
$L_{corr} = \infty$



$L_{corr} = 3.14$



$L_{corr} = 1.74$



$L_{corr} = 0.31$