Spectrum of the incompressible viscous Rayleigh-Taylor system

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SMAI, June 21st, 2021

Navier-Stokes equation

- We will study the case of a viscous fluid as well as revisit results on the inviscid situation. And our main task is to study the spectral analysis of the linearized system of equations.
- The incompressible Navier-Stokes equations in the presence of gravitational field in R^3 read as

$$\begin{cases} \partial_t \rho + \text{div}(\rho \vec{u}) = 0, \\ \partial_t(\rho \vec{u}) + \text{div}(\rho \vec{u} \otimes \vec{u} + p) = \mu \Delta \vec{u} - g\rho \vec{e}_3, \\ \text{div}\vec{u} = 0. \end{cases} \tag{1}$$

Here, $(\rho, \vec{u}, p)(t, x_1, x_2, x_3)$ are the density, the velocity, the pressure of the fluid, respectively. $\mu > 0$ is the viscosity coefficient and $g > 0$ is the gravitational constant. The initial conditions are $(\rho, \vec{u})(0, x)$ satisfying the compatibility condition $\text{div}\vec{u}(0, x) = 0$.

- We study the linearized equation around the laminar flow

$$(\rho_0(x_3), \vec{0}, p_0(x_3)),$$

with $p_0' = -g\rho_0$. The prime here denotes the derivative in x_3.

Denote

$$\sigma = \rho - \rho_0, \quad \vec{u} = \vec{u} - \vec{0}, \quad q = p - p_0.$$

The linearized equations read as

$$\begin{cases} \partial_t \sigma + \rho_0' u_3 = 0, \\ \rho_0 \partial_t \vec{u} + \nabla q = \mu \Delta \vec{u} - g\sigma \vec{e_3}, \\ \mathrm{div} \vec{u} = 0. \end{cases} \tag{2}$$

The linear instability study amounts to looking at growing modes of Eq. (2) of the form

$$(\sigma, u_1, u_2, u_3, q)(t, x_1, x_2, x_3) = e^{\lambda t + i(k_1 x_1 + k_2 x_2)}(\theta, v_1, v_2, \psi, r)(x_3),$$

with $(k_1, k_2) \in \mathrm{R}^2$ being the horizontally spatial frequency, $\lambda = \lambda(k_1, k_2)$ being a complex number and $Re\lambda > 0$. That λ will be called the growth rate.

Growing mode solutions

We have the following system

$$\begin{cases} \lambda\theta + \rho_0'\psi = 0, \\ \lambda\rho_0 v_1 + ik_1 r = \mu(v_1'' - (k_1^2 + k_2^2)v_1), \\ \lambda\rho_0 v_2 + ik_2 r = \mu(v_2'' - (k_1^2 + k_2^2)v_2), \\ \lambda\rho_0\psi + r' = \mu(\psi'' - (k_1^2 + k_2^2)\psi) - g\theta, \\ ik_1 v_1 + ik_2 v_2 + \psi' = 0. \end{cases} \tag{3}$$

We aim at seeking regular solutions in $H^s(\mathrm{R})(s \geqslant 1)$ of (3) satisfying that v_1, v_2 and ψ go to 0 as $|x_3| \to \infty$.

Let $\mathcal{R} \in SO(2)$ be the rotation operator such that $\mathcal{R}(k_1, k_2) = (k, 0)$ with $k = \sqrt{k_1^2 + k_2^2} > 0$. k will be called the wave number. If $(\theta, v_1, v_2, \psi, r)$ is a solution of (3) with frequency (k_1, k_2), so that $(\theta, \mathcal{R}(v_1, v_2), \psi, r)$ is also a solution of (3) with frequency $\mathcal{R}(k_1, k_2)$. We move to solve

$$\begin{cases} \lambda\theta + \rho_0'\psi = 0, \\ \lambda\rho_0 v_1 + ikr = \mu(v_1'' - k^2 v_1), \\ \lambda\rho_0 v_2 = \mu(v_2'' - k^2 v_2), \\ \lambda\rho_0\psi + r' = \mu(\psi'' - k^2\psi) - g\theta, \\ ikv_1 + \psi' = 0. \end{cases} \tag{4}$$

Multiplying $(4)_3$ by $\overline{v}_2$ and then using the integration by parts, we have

$$Re\lambda \int_{\mathrm{R}} \rho_0 |v_2|^2 dx_3 = -\mu \int_{\mathrm{R}} (|v_2'|^2 + k^2|v_2|^2) dx_3 \leqslant 0.$$

That yields $v_2 \equiv 0$. Meanwhile, θ, v_1 and r are determined by ψ as

$$v_1 = -\frac{\psi'}{ik}, \quad \theta = -\frac{\rho_0'\psi}{\lambda}, \quad r = \frac{1}{ik}(\mu(v_1'' - k^2 v_1) - \lambda\rho_0 v_1). \tag{5}$$

We deduce a fourth-order ODE on ψ, that is

$$\lambda^2(k^2\rho_0\psi - (\rho_0\psi')') + \lambda\mu(\psi^{(4)} - 2k^2\psi'' + k^4\psi) = gk^2\rho_0'\psi, \tag{6}$$

with the limits

$$\lim_{|x_3|\to\infty} \psi(x_3) = \lim_{|x_3|\to\infty} \psi'(x_3) = 0.$$

Once we have (λ, ψ), then (v_1, v_2, θ, r), we take the inverse rotation operator $\mathcal{R}^{-1}$ to get a solution of (3).

Without viscosity effect, i.e., $\mu = 0$, Eq. (1) is the Euler equation,

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \vec{u}) = 0, \\ \partial_t(\rho \vec{u}) + \mathrm{div}(\rho \vec{u} \otimes \vec{u} + p) = -g\rho \vec{e_3}, \\ \mathrm{div}\vec{u} = 0. \end{cases} \tag{7}$$

System (3) becomes

$$\begin{cases} \lambda\theta + \rho_0'\psi = 0, \\ \lambda\rho_0 v_1 + ik_1 r = 0, \\ \lambda\rho_0 v_2 + ik_2 r = 0, \\ \lambda\rho_0\psi + r' = -g\theta, \\ ik_1 v_1 + ik_2 v_2 + \psi' = 0. \end{cases} \tag{8}$$

Eq. (6) reduces to the following second order ODE,

$$\lambda^2(k^2\rho_0\psi - (\rho_0\psi')') = gk^2\rho_0'\psi, \tag{9}$$

with the limits $\lim_{|x_3|\to\infty} \psi(x_3) = 0$.

Known results

$$\lambda^2(k^2\rho_0\psi - (\rho_0\psi')') = gk^2\rho_0'\psi, \quad \text{with the limits} \lim_{|x_3|\to\infty} \psi(x_3) = 0.$$

- The linear instability of (7) was first studied by Lord Rayleigh (1884) and then by Taylor (1950) (so-called Rayleigh-Taylor instability), assuming that

$$\rho_0(x_3) = \rho_+ 1_{\{x_3>0\}} + \rho_- 1_{\{x_3<0\}} \quad (\rho_+ > \rho_- > 0). \tag{10}$$

The authors prove that there is a unique growth rate $\lambda_0 = \sqrt{gk\frac{\rho_+ - \rho_-}{\rho_+ + \rho_-}}$ such that (9) has a unique family of solutions, spanned by $\psi_0 \in H^1(\mathrm{R})$, with $\psi_0(0) > 0$ and $\|\psi_0\|_{H^1(\mathrm{R})} = 1$ as $\lambda = \lambda_0$.

- After that, the Rayleigh-Taylor instability, linear and even nonlinear one in $H^s(s \geqslant 3)$ is given thanks to Helffer-Lafitte ('03), Guo-Hwang ('03) for continuous profile ρ_0 satisfying

$$\rho_0 \in C^\infty(\mathrm{R}), \quad \rho_0' \geqslant 0, \quad \lim_{x_3\to\pm\infty} \rho_0(x_3) = \rho_\pm \in (0, +\infty). \tag{11}$$

Aim

$$\lambda^2(k^2\rho_0\psi - (\rho_0\psi')') + \lambda\mu(\psi^{(4)} - 2k^2\psi'' + k^4\psi) = gk^2\rho_0'\psi,$$

with the limits

$$\lim_{|x_3|\to\infty} \psi(x_3) = \lim_{|x_3|\to\infty} \psi'(x_3) = 0.$$

With viscosity effect, of purpose is the viscous study of the linear Rayleigh-Taylor instability to the following cases of density profile ρ_0,

1. ρ_0 is piecewise constant,
2. ρ_0' is compactly supported,
3. ρ_0' is non compactly supported and positive everywhere.

The piecewise constant profile

Let us consider a piecewise constant profile, as stated in (10),

$$\rho_0(x_3) = \rho_+ 1_{\{x_3>0\}} + \rho_- 1_{\{x_3<0\}} \quad (\rho_+ > \rho_- > 0).$$

The multiplication operator ρ_0' will be seen as the Dirac measure $(\rho_+ - \rho_-)\delta_0$. Then, Eq.

$$\lambda^2(k^2\rho_0\psi - (\rho_0\psi')') + \lambda\mu(\psi^{(4)} - 2k^2\psi'' + k^4\psi) = gk^2\rho_0'\psi$$

rewrites as

$$\lambda^2(k^2\rho_0\psi - (\rho_0\psi')') + \lambda\mu(\psi^{(4)} - 2k^2\psi'' + k^4\psi) = gk^2(\rho_+ - \rho_-)\psi(0). \quad (12)$$

We generalize the classical result of Rayleigh-Taylor by the following theorem.

Theorem (Lafitte-N., '20)

As μ being sufficiently small, Eq. (12) admits a unique growth rate λ_μ such that $(12)_{\lambda=\lambda_\mu}$ has a unique solution $\psi_\mu \in H^1(\mathrm{R})$, with $\psi_\mu(0) > 0$ and $\|\psi_\mu\|_{H^1(\mathrm{R})} = 1$. In addition, there holds

$$\lim_{\mu\to 0} \lambda_\mu = \lambda_0, \quad \lim_{\mu\to 0} \psi_\mu = \psi_0 \quad \textit{strongly in } H^1(\mathrm{R}). \quad (13)$$

Smooth increasing profile

While, the growth rate is approximately equal to $\sqrt{gk\frac{\rho_+ - \rho_-}{\rho_+ + \rho_-}}$ for piecewise constant.

Lemma

For smooth increasing profile, we necessarily have $0 < \lambda^2 \leqslant g \max_{\mathbb{R}} \frac{\rho_0'}{\rho_0}$.

Two kinds of profile will be considered.

1. ρ_0' is compactly supported,
2. ρ_0' is non compactly supported and positive everywhere.

The investigation for both profile shares the same line. We divide R into three regions $(-\infty, x_-), (x_-, x_+)$ and $(x_+, +\infty)$. After finding solutions on each region, we then match them.

The compactly supported profile

- We begin with the compactly supported profile, i.e. ρ_0 satisfies

$$\rho_0' \in C_0^1(\mathrm{R}), \quad \mathrm{supp}\rho_0' = (-a, a), \quad \rho_0(x_3) = \rho_+(\text{or } \rho_-) \text{ as } x_3 > a(\text{or } x_3 < -a). \tag{14}$$

- On $(-\infty, -a)$, Eq.

$$\lambda^2(k^2\rho_0\psi - (\rho_0\psi')') + \lambda\mu(\psi^{(4)} - 2k^2\psi'' + k^4\psi) = gk^2\rho_0'\psi$$

becomes

$$\lambda^2(k^2\rho_-\psi - (\rho_-\psi')') + \lambda\mu(\psi^{(4)} - 2k^2\psi'' + k^4\psi) = 0, \tag{15}$$

that admits bounded solutions at $-\infty$, that are

$$\psi_-(x_3) = A_1^- e^{kx_3} + A_2^- e^{\tau_- x_3}, \quad \tau_- = \sqrt{k^2 + \lambda\rho_-/\mu},$$

Similarly, we have bounded solution at $+\infty$, that are

$$\psi_+(x_3) = A_1^+ e^{-kx_3} + A_2^+ e^{-\tau_+ x_3}, \quad \tau_+ = -\sqrt{k^2 + \lambda\rho_+/\mu}.$$

- We are left to solve (6) on $(-a, a)$. What are the suitable boundary conditions for (6) on $(-a, a)$?

- At $x_3 = -a$, $(\psi, \psi', \psi'', \psi''')^T(-a)$ belongs to the space spanned by two vectors $(1, k, k^2, k^3)^T$ and $(1, \tau_-, \tau_-^2, \tau_-^3)^T$ and at $x_3 = a$, $(\psi, \psi', \psi'', \psi''')^T(a)$ belongs to the space spanned by two vectors $(1, -k, k^2, -k^3)^T$ and $(1, -\tau_+, \tau_-^2, -\tau_+^3)^T$.
- That imposes the following boundary conditions at $x_3 = -a$,

$$\begin{cases} k\tau_-\psi(-a) - (k+\tau_-)\psi'(-a) + \psi''(-a) = 0, \\ k\tau_-(k+\tau_-)\psi(-a) - (k^2 + k\tau_- + \tau_-^2)\psi'(-a) + \psi'''(-a) = 0. \end{cases} \tag{16}$$

and at $x_3 = a$,

$$\begin{cases} k\tau_+\psi(a) + (k+\tau_+)\psi'(a) + \psi''(a) = 0, \\ -k\tau_+(k+\tau_+)\psi(a) - (k^2 + k\tau_+ + \tau_+^2)\psi'(a) + \psi'''(a) = 0. \end{cases} \tag{17}$$

of Eq. (6),

$$\lambda^2(k^2\rho_0\psi - (\rho_0\psi')') + \lambda\mu(\psi^{(4)} - 2k^2\psi'' + k^4\psi) = gk^2\rho_0'\psi.$$

The compactly supported profile

We then solve (6)-(16)-(17) by using the spectral theory of a compact and self-adjoint operator for a Sturm-Liouville problem on the finite interval $(-a, a)$.

Theorem (Lafitte-N., '20)

There exists an infinite sequence of growth rate $(\lambda_n)_{n\geq 1}$ decreasing towards to 0 such that (6) with parameter $\lambda = \lambda_n$ has a solution $\psi_n \in H^4(\mathbb{R})$.

The strictly increasing profile

We then consider a strictly increasing profile of density, i.e. $\rho_0' > 0$, that satisfies

$$\rho_0 \in C^2(\mathrm{R}), \qquad \lim_{x_3 \to \pm\infty} \rho_0(x_3) = \rho_\pm \text{ are finite.}$$

The illustration follows the same line of the compactly supported profile. In the vicinity of $\pm\infty$, we rewrite (6) as a system of ODEs

$$U'(x_3) = (L(x_3,\lambda) + \rho_0'(x_3)R(\lambda))U(x_3), \tag{18}$$

where

$$L(x_3,\lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\lambda k^2 \rho_0(x_3) + \mu k^4}{\mu} & 0 & \frac{\lambda \rho_0(x_3)}{\mu} + 2k^2 & 0 \end{pmatrix}$$

and

$$R(\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{gk^2}{\lambda\mu} & \frac{\lambda}{\mu} & 0 & 0 \end{pmatrix}.$$

The non compactly supported profile

- The eigenvalues of L are $\pm k$ and $\pm\sqrt{k^2+\lambda\rho_0(x_3)/\mu}$, that are separated as $\lambda \geqslant \epsilon_\star > 0$.
- Since L is diagonalizable, following Coddington-Levinson, there exists $x_+(\lambda) > 0$ such that we obtain two fundamental bounded solutions $U_{1,2}^+$ of (18) on $(x_+, +\infty)$ satisfying

$$\|U_1^+(x_3)\| \approx e^{-kx_3}, \quad \|U_2^+(x_3)\| \approx e^{-\tau_+ x_3} \quad (\tau_+ = \sqrt{k^2+\lambda\rho_+/\mu}). \quad (19)$$

- Similarly, there exists $x_-(\lambda) < 0$ such that we obtain two fundamental bounded solutions $U_{1,2}^-$ of (18) on $(-\infty, x_-)$ satisfying

$$\|U_1^-(x_3)\| \approx e^{kx_3}, \quad \|U_2^-(x_3)\| \approx e^{\tau_- x_3} \quad (\tau_- = \sqrt{k^2+\lambda\rho_-/\mu}). \quad (20)$$

The non compactly supported profile

- We go back to (6) to obtain two fundamental bounded solutions $\psi_{1,2}^{+}$ of (6) on $(x_+, +\infty)$ satisfying

$$\psi_1^+(x_3) \approx e^{-kx_3}, \quad \psi_2^+(x_3) \approx e^{-\tau_+ x_3} \tag{21}$$

and two fundamental bounded solutions $\psi_{1,2}^{-}$ of (6) on $(-\infty, x_-)$ satisfying

$$\psi_1^-(x_3) \approx e^{kx_3}, \quad \psi_2^-(x_3) \approx e^{\tau_- x_3}. \tag{22}$$

- The boundary conditions are then found to match with $\psi_{1,2}^{\pm}$ at $x_\pm$. Following the arguments on the case of compactly supported profile, we obtain our third theorem.

Theorem (Lafitte-N., '20)

Let $\epsilon_\star > 0$ be given. There exists a finite number $N(\epsilon_\star)$ of growth rate $(\lambda_n)_{n\geq 1}$ bounded below by $\epsilon_\star$ such that (6) with parameter $\lambda = \lambda_n$ has a solution $\psi_n \in H^4(\mathbb{R})$. Furthermore, $N(\epsilon_\star) \to +\infty$ as $\epsilon_\star \to 0$.

Conclusions

$$\lambda^2(k^2\rho_0\psi - (\rho_0\psi')') + \lambda\mu(\psi^{(4)} - 2k^2\psi'' + k^4\psi) = gk^2\rho_0'\psi,$$

with the limits

$$\lim_{|x_3|\to\infty} \psi(x_3) = \lim_{|x_3|\to\infty} \psi'(x_3) = 0.$$

1. There exists a **unique** growth rate if ρ_0 is piecewise constant.
2. There exists an "**infinite**" sequence of growth rate decreasing towards to 0 if $\rho_0 \in C^2(\mathrm{R})$.

Perspectives

1. Reduce the condition $\lambda \geqslant \epsilon_\star$,
2. Towards to the nonlinear instability of $(\rho_0(x_3), \vec{0}, p_0(x_3))$ in the viscous case.
3. Continue using this method to some extended cases of Navier-Stokes equation with gravitational force field.

References

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Thank you for your attention