Intermittency in Lagrangian stochastic models for turbulent flows: genuine characterization and design of a versatile numerical approach

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Collaboration with Alexandre Richard [2], Rémi Zamansky [4], Aymeric Vié [1,2], Marc Massot [3]

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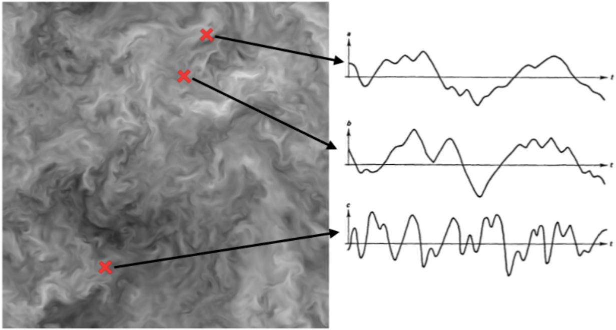
A picture of turbulence



The Navier-Stokes equations :

$$\begin{cases} \nabla \boldsymbol{u} = 0 \\ \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\dfrac{1}{\rho} \nabla P + \nu \nabla^2 \boldsymbol{u} \end{cases}$$

Non-linear and non-local PDE

Simulation computationally expensive:

- Large range of scales
- Chaotic system $\rightarrow$ several realizations

Statistical description

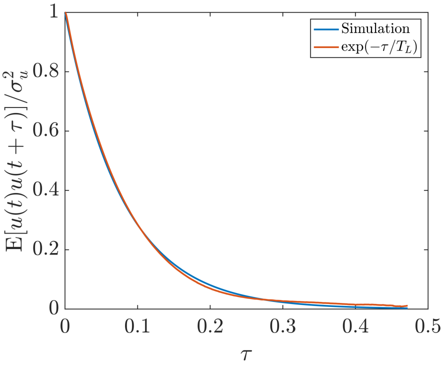
Turbulence can be statistically described with the correlation functions:

$$R_{ij\dots}(\boldsymbol{x}, t; \boldsymbol{x}', t'; \dots) = \mathbb{E}[u_i(\boldsymbol{x}, t) u_j(\boldsymbol{x}', t') \dots]$$

Lagrangian velocity correlation function (G.I. Taylor in 1935: beginning of the modern statistical approach):

$$\boxed{R_u^L(\tau) = \mathbb{E}[u_i(t) u_i(t+\tau)]}$$

The variance is given by: $R_u^L(0) = \sigma_u^2$.



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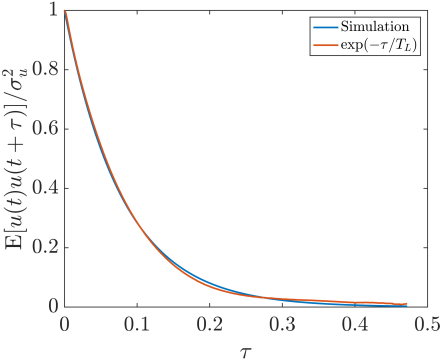
Lagrangian integral time $\sim$ correlation time

$$T_L = \frac{1}{\sigma_u^2} \int_0^\infty R_u^L(\tau) d\tau$$



The similarity hypotheses[1]: K41

Injection of energy

Characteristic time of velocity auto-correlation

Largest scales: T_L

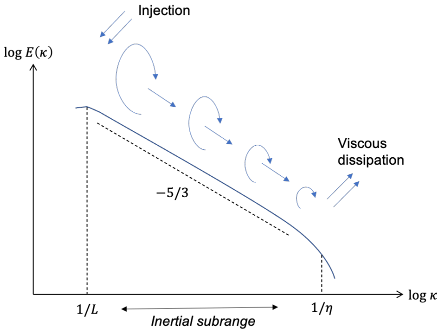
Dissipation of energy

Energy dissipation rate: $\varphi = \nu \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}$

$\rightarrow$ Mean dissipation: $\langle\varphi\rangle$

Smallest scales: $\tau_\eta \equiv (\nu/\langle\varphi\rangle)^{1/2}$

Kolmogorov 1941: Turbulence is universal in the inertial range. Velocity increments $\Delta_\tau u = u(t+\tau) - u(t)$ depend on the mean dissipation.



Similarity hypothesis

For

$$\tau_\eta \ll \tau \ll T_L, \quad \mathbb{E}[(\Delta_\tau u)^p] = f(\tau, p, \langle\varphi\rangle)$$

[1] Kolmogorov1941.

The refined similarity hypotheses[2]

Similarity hypothesis

For
$\tau_\eta \ll \tau \ll T_L, \quad \mathbb{E}[[\Delta_\tau u]^p] = f(\tau, p, \langle\varphi\rangle)$

Confirmed experimentally for the

- Second-order moment:
$$\mathbb{E}[[\Delta_\tau u]^2] \sim \langle\varphi\rangle\tau$$
$\rightarrow -5/3$ slope in the energy spectrum
- Third-order moment:
$$\mathbb{E}[[\Delta_\tau u]^3] \sim (\langle\varphi\rangle\tau)^{3/2}$$
$\rightarrow 4/5$-law

The refined similarity hypotheses[2]

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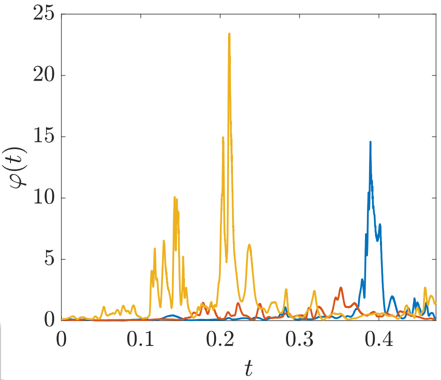
$$\mathbb{E}[[\Delta_\tau u]^3] \sim (\langle\varphi\rangle\tau)^{3/2}$$

$\rightarrow$ 4/5-law

BUT, no generalization to higher moments:

Refined similarity hypotheses

$$\mathbb{E}[[\Delta_\tau u]^p] \not\sim (\langle\varphi\rangle\tau)^{p/2} \rightarrow f(\tau, p, \varphi)$$



$\rightarrow$ Large fluctuations of φ = intermittency

Objective

A first definition of Intermittency

- Strongly fluctuent, sudden and brief high activity.
- In turbulence: energy cascade with large fluctuations in the transfer of energy between eddies of different scales
- Statistical consequences: extreme events are more likely than for a Gaussian distribution.

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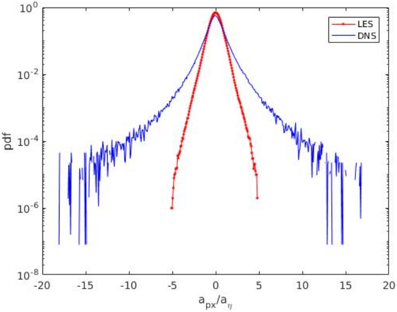
Provide a stochastic model for φ, the dissipation, with intermittent properties.



Previous mathematical results in similar context

- Bacry, Delour, Muzy. A multifractal random walk. (Physical Review E, 2001).
- Chevillard, Robert, Vargas. A stochastic representation of the local structure of turbulence. EPL (Europhysics Letters, 2010).
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$\rightarrow$ **First non Gaussian/non diffusive models.**

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→ **More complex models: fBM, Hurst=0, adapted model, versatile, numerically efficient.**

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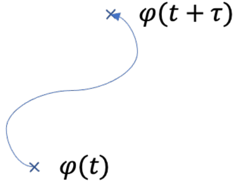
- Abi Jaber, El Euch. Multifactor Approximation of Rough Volatility Models. (SIAM Journal on Financial Mathematics, 2019).

The refined similarity hypotheses[3] : K62

$\mathbb{E}[[\Delta_\tau u]^p] \not\sim (\langle\varphi\rangle\tau)^{p/2} \to \langle\varphi\rangle$ is not the appropriate scale.
Define local scales:

$$\boxed{\text{Locally-averaged dissipation:} \quad \varphi_\tau(t) = \frac{1}{\tau}\int_t^{t+\tau} \varphi(s)\,\mathrm{d}s}$$



Refined similarity hypotheses

- Local scaling:

$$\text{For} \quad \tau_\eta \ll \tau \ll T_L, \quad \begin{aligned} \mathbb{E}[(\Delta_\tau u)^p|\varphi_\tau] &\sim (\varphi_\tau\tau)^{p/2} \\ \mathbb{E}[(\Delta_\tau u)^p] &\sim C_p\tau^{p/2}\mathbb{E}[\varphi_\tau^{p/2}] \end{aligned}$$

- Log-normal distribution of φ_τ, with:

$$\sigma^2_{\log\varphi_\tau} = a + b\log(T_L/\tau)$$

[3] Kolmogorov1962.

Characterization of intermittency

Requirements, in the inertial range: for $\tau_\eta \ll \tau \ll T_L$

(i) Kolmogorov 1962: φ is log-normal with $\sigma^2_{\log \varphi_\tau} \sim \log \frac{T_L}{\tau}$

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(i) Kolmogorov 1962: φ is log-normal with $\sigma^2_{\log \varphi_\tau} \sim \log \dfrac{T_L}{\tau}$

(ii) Power-law correlation for the locally-averaged dissipation:
$\mathbb{E}[\varphi_\tau^p] \sim \left(\dfrac{T_L}{\tau}\right)^{K(p)}$, where $K(p)$ is a non-linear, convex function.

Remark, in the dissipative range: for $\tau \ll \tau_\eta$

In real turbulence, there is supplementary physical interpretation with dissipation: for $\varphi_\tau = \varphi_{\tau_\eta} = \varphi$ and

(iii) $\sigma^2_{\log \varphi} \sim \log \dfrac{T_L}{\tau_\eta}$

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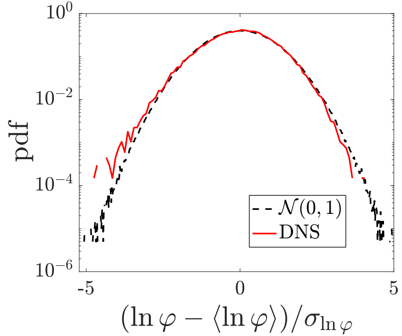
Models

Multifractal fields: Discrete cascade models, Continuous multiplicative cascades, Eulerian, Lagrangian etc...

Gaussian Multiplicative Cascade

We are looking for a time stochastic process

$$\boxed{\varphi(t) = \langle\varphi\rangle \exp(\chi_t)}$$ where χ_t is Gaussian.



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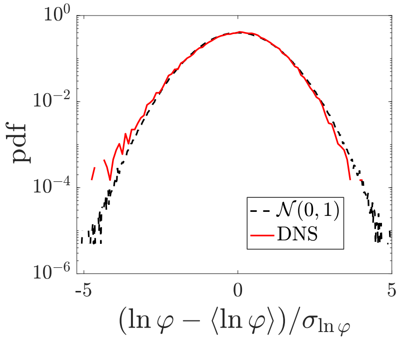
Parametrization by a zero-average Gaussian process X_t: $\chi_t = \sqrt{\mu^\ell} X_t - \frac{\mu^\ell}{2}\mathbb{E}[X_t^2]$



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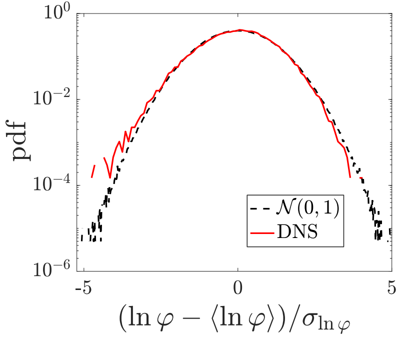
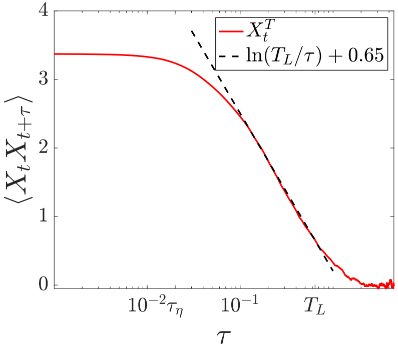
With X_t a (approximated) log-correlated process: $\mathbb{E}[X_t X_{t+\tau}] = \log \frac{T_L}{\tau} + g(t, \tau)$





Modeling intermittent pseudo-dissipation

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Requirements

(i) φ is log-normal with $\sigma^2_{\log \varphi_\tau} \sim \log \dfrac{T_L}{\tau}$

(ii) for $\tau_\eta \ll \tau \ll T_L$, $\mathbb{E}[\varphi_\tau^p] \sim \left(\dfrac{T_L}{\tau} \right)^{K(p)}$

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We can show that

- $\mathbb{E}[X_t X_{t+\tau}] \sim \log \frac{T_L}{\tau} \Rightarrow$ (ii)
- $\mathbb{E}[X_t^2] \sim \log \frac{T_L}{\tau_\eta} \Rightarrow$ (iii)

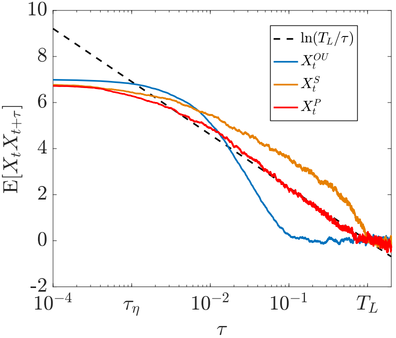
In that case, $K(p) = \frac{\mu^\ell}{2} p(p-1)$

Introduction to log-correlated processes

Ornstein-Uhlenbeck process[1]

$$\mathrm{d}X_t = -\frac{1}{T_\chi} X_t \,\mathrm{d}t + \left(2\frac{\sigma_\chi^2}{\mu^\ell T_\chi}\right)^{1/2} \mathrm{d}W_t$$

Exponential decay of the autocorrelation:
$\mathbb{E}[X_t X_{t+\tau}] \sim \mathrm{e}^{-\tau/T_\chi}$



[1] Pope1990
[2] Pereira2018

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Fractional Ornstein-Uhlenbeck process[2]

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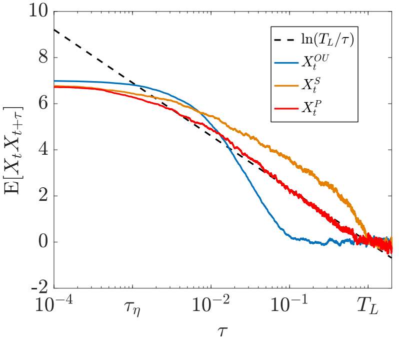

Fractional Brownian motion:

$$W_t^H = \frac{1}{\Gamma(H+1/2)} \int_0^t (t-s)^{H-1/2} \,\mathrm{d}W_s, \quad H > 0$$

[1] Pope1990
[2] Pereira2018

Fractional Brownian motion

Replace Brownian motion W_t by a **fractional Brownian motion** W_t^H, a non-Markovian process.

Fractional Brownian motion

Generalization of Brownian Motion: increments are not independent.

- Continuous Gaussian process of zero mean
- Covariance function:

$$\mathbb{E}[W_t^H W_{t+s}^H] = \frac{1}{2}(|t|^{2H} + |t+s|^{2H} - |s|^{2H})$$

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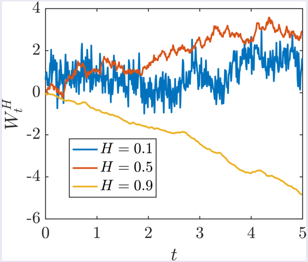

$\rightarrow$ fractional Ornstein-Uhlenbeck: $dX_t = \theta(X_t - \mu)dt + dW_t^H$

Fractional Brownian motion

Regularized fractional Brownian motion, $H > 0$, $\tau_\eta > 0$:

$$W_t^H = \frac{1}{\Gamma(H+1/2)} \int_0^t (t-s)^{H-1/2}\,\mathrm{d}W_s, \quad \text{or} \quad W_t^{\tau_\eta} = \frac{1}{\sqrt{\pi}} \int_0^t (t-s+\tau_\eta)^{-1/2}\,\mathrm{d}W_s,$$

$\rightarrow$ Regularized fractional Brownian motion with stationary increments

$$\begin{aligned}
\overline{W_t^H} &= \frac{1}{\Gamma(H+1/2)} \left\{ \int_{-\infty}^0 \left[(t-s)^{H-1/2} - (-s)^{H-1/2} \right] \mathrm{d}W_s + \int_0^t (t-s)^{H-1/2}\,\mathrm{d}W_s \right\} \\
\overline{W_t^{\tau_\eta}} &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^t \left((t-s+\tau_\eta)^{-1/2} - (-s+\tau_\eta)_+^{-1/2} \right) \mathrm{d}W_s \\
\mathrm{d}\overline{W_t^{\tau_\eta}} &= -\frac{1}{2} \frac{1}{\sqrt{\pi}} \int_{-\infty}^t (t-s+\tau_\eta)^{-3/2}\,\mathrm{d}W_s\,\mathrm{d}t + (\pi\tau_\eta)^{-1/2}\,\mathrm{d}W_t
\end{aligned} \tag{2.1}$$

Pereira et al.[3] define:

$$\mathrm{d}X_t^P = -\frac{1}{T_L} X_t^P\,\mathrm{d}t + \sqrt{\pi}\,\mathrm{d}\overline{W_t^{\tau_\eta}}$$

Schmitt[4] defines:

$$X_t^S = \int_{t+\tau_\eta-T_L}^t (t-s+\tau_\eta)^{-1/2}\,\mathrm{d}W_s$$

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Main ingredient for X_t

Regularized fractional Gaussian Noise of Hurst $H = 0$

... But simulation of $W_t^{\tau_\eta}$ is computationally expensive.

[3] Pereira2018
[4] Schmitt2003

Approximation of a fBM by Ornstein-Uhlenbeck processes

Introduce a generic representation of all these processes in a unified framework:
Remark that the Laplace Transform of $x^{-1/2}$ is given by :

$$\mathcal{L}\left\{\frac{1}{\sqrt{x}}\right\}(s) = \int_0^{+\infty} \frac{1}{\sqrt{x}} e^{-sx}\,\mathrm{d}x = \sqrt{\pi} s^{-1/2}$$

$$W_t^{\tau_\eta} = \frac{1}{\sqrt{\pi}} \int_0^t (t-s+\tau_\eta)^{-1/2}\,\mathrm{d}W_s = \frac{1}{\sqrt{\pi}} \int_0^t \int_0^{+\infty} \frac{1}{\sqrt{\pi x}} e^{-(t-s+\tau_\eta)x}\,\mathrm{d}x\ \ \mathrm{d}W_s.$$

But, using stochastic Fubini theorem, we can write for all $\tau_\eta > 0$

$$W_t^{\tau_\eta} = \int_0^{+\infty} \frac{1}{\pi\sqrt{x}} e^{-\tau_\eta x} \left(\int_0^t e^{-x(t-s)}\,\mathrm{d}W_s\right)\mathrm{d}x = \int_0^{+\infty} \frac{1}{\pi\sqrt{x}} e^{-\tau_\eta x} Y_t^x\,\mathrm{d}x$$

where $(Y_t^x)_{x\in\mathbb{R}}$ is a family of Ornstein-Uhlenbeck processes, such that for all $x \in \mathbb{R}$

$$\begin{cases} Y_0^x & = 0 \\ \mathrm{d}Y_t^x & = -xY_t^x\,\mathrm{d}t + \mathrm{d}W_t \end{cases}$$

A Framework encompassing existing processes

We can show that :

$$W_t^{\tau_\eta} = \int_0^\infty \phi_{t,x}(W) k_{\tau_\eta}(x)\,\mathrm{d}x$$

$W_t^{\tau_\eta}$	$\frac{1}{\sqrt{\pi}} \int_0^t (t-s+\tau_\eta)^{-1/2}\,\mathrm{d}W_s$	$\int_0^\infty Y_t^x \frac{e^{-\tau_\eta x}}{\pi\sqrt{x}}\,\mathrm{d}x$

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$W_t^{\tau_\eta}$	$\frac{1}{\sqrt{\pi}} \int_0^t (t-s+\tau_\eta)^{-1/2}\,\mathrm{d}W_s$	$\int_0^\infty Y_t^x \frac{e^{-\tau_\eta x}}{\pi\sqrt{x}}\,\mathrm{d}x$
$\overline{W_t^{\tau_\eta}}$		$\int_0^\infty (\overline{Y_t^x} - Y_0^x) \frac{e^{-\tau_\eta x}}{\pi\sqrt{x}}\,\mathrm{d}x$

A Framework encompassing existing processes

We can show that :

$$W_t^{\tau_\eta} = \int_0^\infty \phi_{t,x}(W) k_{\tau_\eta}(x) \, \mathrm{d}x$$

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W_t^H	$\frac{1}{\Gamma(H+1/2)} \int_0^t (t-s)^{H-1/2}\, \mathrm{d}W_s$	$\int_0^\infty Y_t^x \frac{\cos(\pi H)}{\pi x^H \sqrt{x}}\, \mathrm{d}x$

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$$\mathrm{d}X_t = -\frac{1}{T_L} X_t \, \mathrm{d}t + \sqrt{\pi} \, \mathrm{d}\overline{W_t^{\tau_\eta}}$$

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Assume $X_t = \int_0^\infty \phi_{t,x}(W) k_{\tau_\eta}(x)\,\mathrm{d}x$ and find a regularization that ensures:

- Existence of X_t
  $\phi_{t,x}(W)$ is almost surely $k_{\tau_\eta}(x)\,\mathrm{d}x$ integrable
- Logarithmic autocorrelation $\mathbb{E}[X_t X_{t+\tau}]$ in inertial subrange:
  $\to k_{\tau_\eta}(x) \sim x^{-1/2}$
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$$\boxed{X_t = \int_0^{\infty} \phi_{t,x}(W) k_{\tau_\eta}(x)\,\mathrm{d}x}$$

Regularization for:

- existence;
- logarithmic behavior;
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Pope : X_t^{OU}	$\int_0^t \omega e^{-(t-s)/T_\chi}\,\mathrm{d}W_s$	$\int_0^{\infty} Y_t^x \omega \delta(x - T_\chi^{-1})\,\mathrm{d}x$

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Pereira : X_t^P	$\sqrt{\pi} \int_{-\infty}^t \mathrm{e}^{-(t-s)/T_L} \overline{\mathrm{d}W_s^{\tau_\eta}}$	$\int_0^\infty \left(\int_{-\infty}^t \mathrm{e}^{-(t-s)/T_L} \overline{\mathrm{d}Y_s^x} \right) \frac{\mathrm{e}^{-\tau_\eta x}}{\sqrt{\pi x}}\,\mathrm{d}x$

A new proposition for X_t [1]

$$X_t = \int_0^\infty \phi_{t,x}(W) k_{\tau_\eta}(x)\,\mathrm{d}x$$

$$\boxed{X_t^{new} = \int_0^\infty \overline{Y_t^x} \frac{g_{T_L}(x) - g_{\tau_\eta}(x)}{\sqrt{x}}\,\mathrm{d}x}$$

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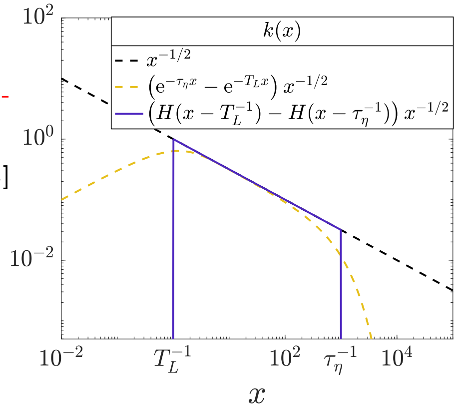

[1] Letournel R., Goudenège L, Zamansky R. et al. Revisiting the framework for intermittency in Lagrangian stochastic models for turbulent flows: a way to an original and versatile numerical approach. Physical Review E. (accepted in june 2021)

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X_t^{new}	$\int_{-\infty}^t (t-s+\tau_\eta)^{-1/2} - (t-s+T_L)^{-1/2}\,\mathrm{d}W_s$	$\int_0^\infty \overline{Y_t^x} \frac{g_{T_L}(x) - g_{\tau_\eta}(x)}{\sqrt{\pi x}}\,\mathrm{d}x$

Limit process as $\tau_\eta \to 0$

Be careful that $(X_t^{new}, X_t^S, X_t^P)$ are all defined with regularization kernels $(k_{\tau_\eta})_{\tau_\eta > 0}$. Thanks to this regularization, for all $\tau_\eta > 0$, the processes $(X_t^{(\tau_\eta)})_{t\in[0,T]}$ are Gaussian and stationary. Their laws are completely determined by their covariance functions

$$C_{\tau_\eta}^{new,S,P}(t,s) := \mathbb{E}[X_t^{(\tau_\eta)} X_s^{(\tau_\eta)}].$$

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The family of processes $((X_t^{(\tau_\eta)})_{t\in[0,T]})_{\tau_\eta>0}$ converges weakly in law to Gaussian log-correlated processes $(X_t^0)_{t\in[0,T]}$ with covariance functions

$$C_0(t,s) = \lim_{\tau_\eta \to 0} C_{\tau_\eta}^{new,S,P}(t,s) = \log_+ \frac{1}{|t-s|} + g(t,s)$$

with bounded function $g \in \{g^{new}, g^S, g^P\}$.

Limit process as $\tau_\eta \to 0$

Finally the solution of the SDE $\mathrm{d}u_t^{(\tau_\eta)} = -\frac{1}{T_u} u_t^{(\tau_\eta)} \,\mathrm{d}t + \varphi_{\tau_\eta}(t)\,\mathrm{d}t$ driven by

$$\varphi_{\tau_\eta}(t) := \langle \varphi \rangle \exp(\chi_t^{(\tau_\eta)}) = \langle \varphi \rangle \exp\left(\sqrt{\mu^\ell} X_t^{(\tau_\eta)} - \frac{\mu^\ell}{2} \mathbb{E}[(X_t^{(\tau_\eta)})^2] \right)$$

converges in law (in the distributional sense $\langle u^{(\tau_\eta)}, \phi \rangle_{\mathcal{D}',\mathcal{D}} \longrightarrow_{\tau_\eta \to 0} \langle u, \phi \rangle_{\mathcal{D}',\mathcal{D}}$) to a distributional process u such that

$$u_t = \int_0^t e^{-\frac{(t-s)}{T_u}} \Gamma(\mathrm{d}s) \qquad \left(\sim \lim_{\tau_\eta \to 0} \int_0^t e^{-\frac{(t-s)}{T_u}} \varphi_{\tau_\eta}(s)\,\mathrm{d}s \right)$$

with Γ the universal Gaussian Multiplicative Chaos.

Finite sum of Ornstein-Uhlenbeck processes

$$\boxed{X_t^{new} \equiv \int_0^\infty Y_t^x k_{\tau_\eta}(x)\,\mathrm{d}x \approx X_t^{new,N} \equiv \sum_{i=1}^N \omega_i Y_t^{x_i}}$$

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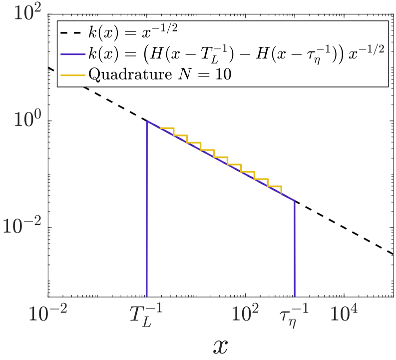
Geometric partition: $x_i = \frac{1}{T_L}\left(\frac{T_L}{\tau_\eta}\right)^{\frac{i-1/2}{N}}$, $\omega_i = \frac{1}{\sqrt{\pi x_i}}\Delta x_i$, for $i = 1, ..., N$



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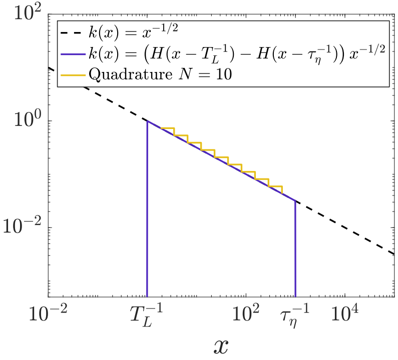
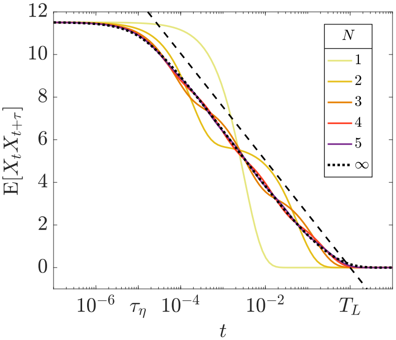




Discussion

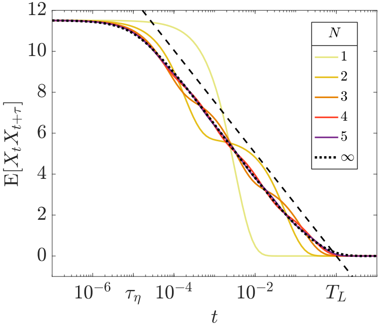
$$X_t^{new} \equiv \int_0^\infty Y_t^x k_r(x)\,\mathrm{d}x \leftrightarrow X_t^{new,N} \equiv \sum_{i=1}^{N} \omega_i Y_t^{x_i}$$

Physical interpretation

- Adaptation of Pope's process for high $\mathrm{Re}_\lambda \sim \frac{T_L}{\tau_\eta}$: extend the covering of the inertial range with evenly-distributed time-scales
- X_t^{new} "continuous" process, no scale

  $\updownarrow$

  $X_t^{new,N}$ "discrete" cascade, representative scales



Discussion

$$X_t^N \equiv \sum_{i=1}^{N} \omega_i Y_t^{x_i}$$

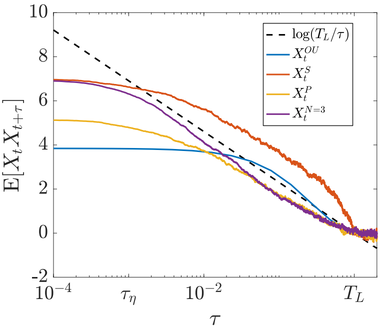
Computational benefits

$$\forall i = 1, \ldots, N \quad \mathrm{d}Y_t^{x_i} = -x_i Y_t^{x_i}\,\mathrm{d}t + \mathrm{d}W_t$$

vs

$$W_t^{\tau_\eta} = \frac{1}{\sqrt{\pi}} \int_0^t (t - s + \tau_\eta)^{-1/2}\,\mathrm{d}W_s$$

- Computational efficiency (Few Ornstein-Uhlenbeck processes)
- Low calculation memory
- Versatile



Conclusion

- Intermittency is missing in classical turbulence models
- Characterization of intermittency and modeling with Gaussian Multiplicative Chaos
- General framework to build stochastic processes with logarithmic correlation
- Quasi explicit computation of approximated covariance function
- Exploitation of these singular processes in SDEs
- Numerical simulation of these processes with finite sum of Ornstein-Uhlenbeck
- Perspectives: computation of (approximated) power law function K, more complex SDEs using these singular processes, speed of convergence of sum of OU...

Goudenège, Letournel, Richard. Intermittency in a stochastic modelling of turbulence (In preparation)

Letournel et al. Revisiting the framework for intermittency in Lagrangian stochastic models for turbulent flows: a way to an original and versatile numerical approach. (Physical Review E). 2021.