Waves in quasiperiodic media

The one-dimensional harmonic case with absorption

Pierre Amenoagbadji Sonia Fliss Patrick Joly

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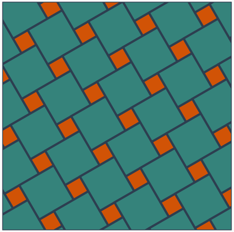

Figure: Periodic tiling

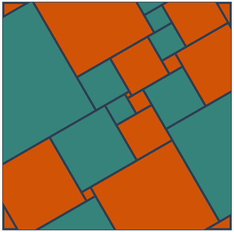

Figure: Random tiling

Physical definition

Quasiperiodic media are **ordered** structures which are **not necessarily periodic**.

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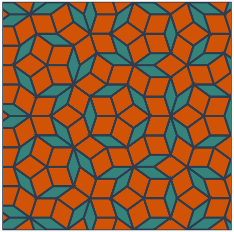


Figure: Quasiperiodic tiling

Figure: Random tiling

Quasiperiodic media

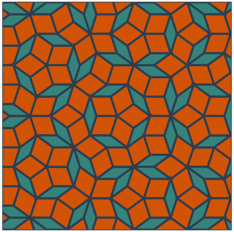
Physical definition

Quasiperiodic media are **ordered** structures which are **not necessarily periodic**.

A physical example: the quasicrystal

First **quasicrystal** formation observed in 1982 by *D. Shechtman*

Figure: Periodic tiling



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1D quasiperiodic functions

Definition (Quasiperiodic medium)

Medium whose physical or geometrical properties can be represented as **quasiperiodic functions**

Definition (Quasiperiodic function of one real variable)

A function $f : \mathbb{R} \to \mathbb{C}$ is said to be **quasiperiodic** of order $n > 0$ if there exist real constants $\delta_1, \ldots, \delta_n$ and a continuous function $F : \mathbb{R}^n \to \mathbb{C}$, $1-$periodic in each variable, such that

$$\forall\, x \in \mathbb{R}, \quad f(x) = F(\delta_1\, x, \ldots, \delta_n\, x).$$

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Remarks

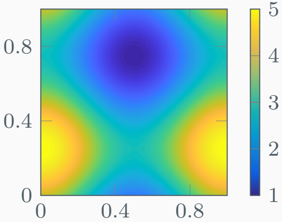
- F is a *periodic extension* of f and $(\delta_1, \ldots, \delta_n)$ is called a *cut direction*
- The pair $(F, (\delta_1, \ldots, \delta_n))$ is not unique
- One can also define **quasiperiodic functions of several real variables**

1D quasiperiodic functions of order 2

Convention

There exists $\theta \in (0, \pi/2)$ and $\mu_p \in \mathscr{C}^0_{per}((0,1)^2)$ such that

$$\mu_\theta(x) = \mu_p(x\,\vec{e}_\theta), \quad \vec{e}_\theta = (\cos\theta, \sin\theta).$$



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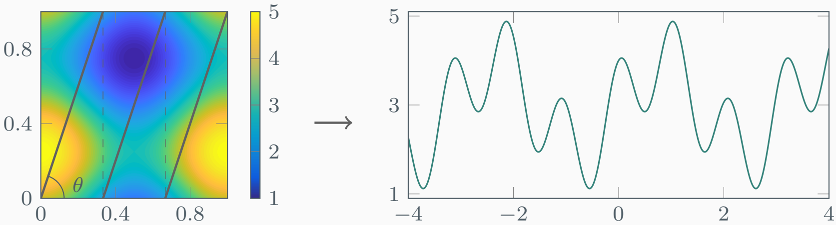

Figure: Trace of a periodic function along $\vec{e}_\theta$ with $\cot\theta = 1/3$ – Rational case

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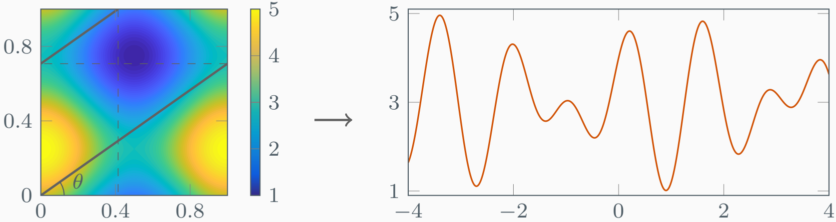

Figure: Trace of a periodic function along $\vec{e}_\theta$ with $\cot\theta = \sqrt{2}$ – Irrational case

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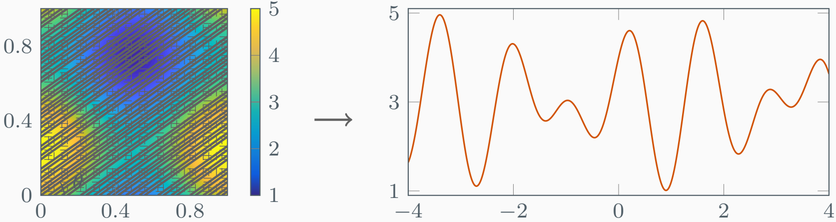

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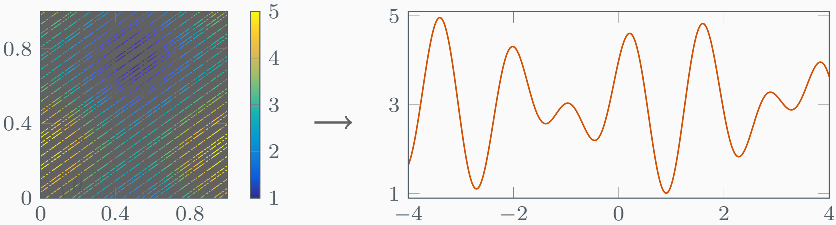

Figure: Trace of a periodic function along $\vec{e}_\theta$ with $\cot\theta = \sqrt{2}$ – Irrational case

PDE with quasiperiodic coefficients

Numerous **theoretical** studies in the context of homogenization

- The medium is submitted to external forces whose length scale are far larger than the characteristic length of the microstructure.
- PDE with rapidly oscillating coefficients such as $-\operatorname{div} A(x/\varepsilon)\nabla u_\varepsilon = f$, where the parameter ε is expected to be small.

Approach for general heterogeneous media

Two-scale and Γ-convergence, almost-periodicity

- Only in the context of homogenization

Braides, 1992
Nguetseng, 2003
Zhikov, *Kozlov*, *Oleinik*, 2012

Very few works in other regimes.

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Cut and project approach

Extend the PDE to a non-elliptic PDE with **periodic** coefficients

- Can be used for problems to which the homogenization theory does not apply

Bouchitté, *Guenneau*, *Zolla*, 2010
Gérard-Varet, *Masmoudi*, 2010
Blanc, *Le Bris*, *Lions*, 2015
Wellander, *Guenneau*, *Cherkaev*, 2019

Very few works in other regimes.

The time-harmonic wave equation

Time-harmonic scalar wave equation

$$-\frac{d}{dx}\left(\mu(x)\,\frac{du}{dx}\right) - \rho(x)\,\omega^2\,u = f(x), \quad \text{in } \mathbb{R}. \qquad (\mathscr{P})$$

Well posedness

- Problem ill-posed in the classical framework

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Computing the physical solution using the limiting absorption principle

1. Add some absorption: $\mathrm{Im}\,(\omega) > 0$
2. Study the solution of $(\mathscr{P})$ as $\mathrm{Im}\,(\omega)$ tends to 0.

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Numerical issue

- How to deal numerically the infinite domain?

The time-harmonic wave equation

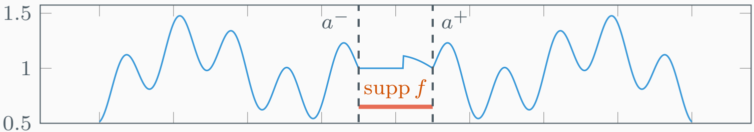
Time-harmonic scalar wave equation **with absorption**

$$-\frac{d}{dx}\left(\mu(x)\,\frac{du}{dx}\right) - \rho(x)\,\omega^2\,u = f(x), \quad \text{in } \mathbb{R}. \qquad (\mathscr{P})$$

Quasiperiodic medium with a local perturbation

There exist $a^- < a^+$ and quasiperiodic functions $\mu_\theta^\pm$ and $\rho_\theta^\pm$ such that

$$\begin{aligned} \mu(x) &= \mu_\theta^-(x - a^-), & \rho(x) &= \rho_\theta^-(x - a^-) & \text{if } x < a^- \\ \mu(x) &= \mu_\theta^+(x - a^+), & \rho(x) &= \rho_\theta^+(x - a^+) & \text{if } x > a^+ \end{aligned} \quad \text{and} \quad \operatorname{supp} f \subset (a^-, a^+)$$



Restriction to a bounded domain

Computations can be restricted to (a^-, a^+) using DtN conditions.

$$-\frac{d}{dx}\left(\mu\,\frac{du_i}{dx}\right) - \rho\,\omega^2\,u_i = f$$

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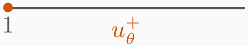
$$\left|\begin{aligned} -\frac{d}{dx}\left(\mu_\theta^-\,\frac{du_\theta^-}{dx}\right) - \rho_\theta^-\,\omega^2\,u_\theta^- &= 0, \quad \mathbb{R}_-^* \\ u_\theta^-(0) &= 1. \end{aligned}\right.$$

$$\left|\begin{aligned} -\frac{d}{dx}\left(\mu_\theta^+\,\frac{du_\theta^+}{dx}\right) - \rho_\theta^+\,\omega^2\,u_\theta^+ &= 0, \quad \mathbb{R}_+^* \\ u_\theta^+(0) &= 1. \end{aligned}\right.$$



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u_i

a^- a^+

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$$-\frac{d}{dx}\left(\mu_\theta^+\,\frac{du_\theta^+}{dx}\right) - \rho_\theta^+\,\omega^2\,u_\theta^+ = 0, \quad \mathbb{R}_+^*$$
$$u_\theta^+(0) = 1.$$

$$\lambda_\theta^- = \mu_\theta^-\,\frac{du_\theta^-}{dx}(0)$$

$$\lambda_\theta^+ = -\mu_\theta^+\,\frac{du_\theta^+}{dx}(0)$$

u_θ^- 1

1 u_θ^+

Restriction to a bounded domain

Computations can be restricted to (a^-, a^+) using DtN conditions.

$u_i(a^-)\, u_\theta^-(x - a^-)$ $\quad$ u_i $\quad$ $u_i(a^+)\, u_\theta^+(x - a^+)$

a^- $\quad$ a^+

$$-\frac{d}{dx}\left(\mu \frac{du_i}{dx}\right) - \rho\, \omega^2\, u_i = f$$

$$\mu \frac{du_i}{dx}(a^-) = \lambda_\theta^- \; u_i(a^-) \qquad \mu \frac{du_i}{dx}(a^+) = \lambda_\theta^+ \; u_i(a^+)$$

$$-\frac{d}{dx}\left(\mu_\theta^- \frac{du_\theta^-}{dx}\right) - \rho_\theta^-\, \omega^2\, u_\theta^- = 0, \quad \mathbb{R}_-^*$$
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$$u_\theta^+(0) = 1.$$

$$\lambda_\theta^- = \mu_\theta^- \frac{du_\theta^-}{dx}(0) \qquad \lambda_\theta^+ = -\mu_\theta^+ \frac{du_\theta^+}{dx}(0)$$

u_θ^- $\quad$ 1 $\quad$ 1 $\quad$ u_θ^+

The Helmholtz equation with quasiperiodic coefficients

Helmholtz equation **with absorption**

$$-\frac{d}{dx}\left(\mu_\theta(x)\,\frac{du_\theta}{dx}\right) - \rho_\theta(x)\,\omega^2\,u_\theta = 0, \quad \text{in } \mathbb{R}_+^*, \quad u_\theta(0) = 1 \qquad (\mathscr{P}_\theta)$$

Quasiperiodic medium

$\exists\ \mu_p, \rho_p : \mathbb{R}^2 \to \mathbb{R}$, $\mu_p, \rho_p \in \mathscr{C}^0_{per}((0,1)^2)$ such that $\mu_\theta(x) = \mu_p(x\,\vec{e}_\theta)$, $\rho_\theta(x) = \rho_p(x\,\vec{e}_\theta)$ with $\vec{e}_\theta = (\cos\theta, \sin\theta)$.

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Well-posedness

$(\mathscr{P}_\theta)$ admits a unique solution $u_\theta \in H^1(\mathbb{R}_+^*)$ (**Lax-Milgram**).

How can one solve $(\mathscr{P}_\theta)$ numerically?

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Description of the cut method

Helmholtz equation with absorption and quasiperiodic coefficients

$$-\frac{d}{dx}\left(\mu_\theta(x)\,\frac{du_\theta}{dx}\right) - \rho_\theta(x)\,\omega^2\,u_\theta = 0, \quad \text{in } \mathbb{R}_+^*, \quad u_\theta(0) = 1 \qquad (\mathscr{P}_\theta)$$

where

$$\operatorname{Im}(\omega) > 0, \quad \mu_\theta(x) = \mu_p(x\,\vec{e}_\theta), \quad \text{and} \quad \rho_\theta(x) = \rho_p(x\,\vec{e}_\theta)$$

The cut method

Seek u_θ as the trace of a two-dimensional function U_θ along the line $\vec{e}_\theta\mathbb{R}_+$, that is,

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Chain rule

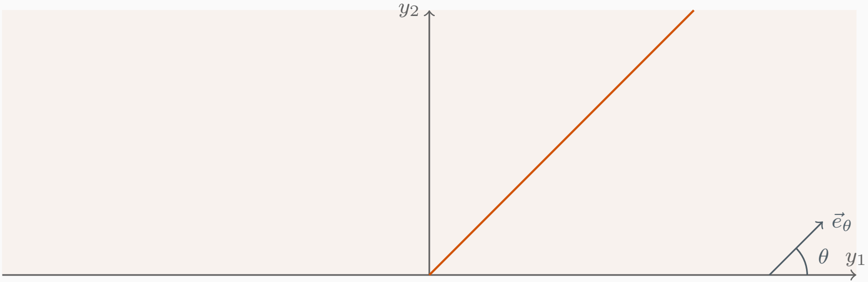
$$\forall\, U : \mathbb{R}^2 \to \mathbb{R}, \quad \frac{d}{dx}[U(x\,\vec{e}_\theta)] = (\vec{e}_\theta \cdot \nabla)U := D_\theta U, \quad D_\theta = \cos\theta\,\partial_{y_1} + \sin\theta\,\partial_{y_2}$$

Extension to a periodic half-plane problem

Trace along $\vec{e}_\theta$ and chain rule

$u_\theta(x) = U_\theta(x\,\vec{e}_\theta)$ and $\dfrac{du_\theta}{dx}(x) = D_\theta U_\theta(x\,\vec{e}_\theta)$

$$-\frac{d}{dx}\left(\mu_\theta\,\frac{du_\theta}{dx}\right) - \rho_\theta\,\omega^2\,u_\theta = 0, \quad \text{in } \mathbb{R}_+^* \longrightarrow -D_\theta\left(\mu_p\,D_\theta U_\theta\right) - \rho_p\,\omega^2\,U_\theta = 0, \quad \text{in } \mathbb{R}\times\mathbb{R}_+^*$$



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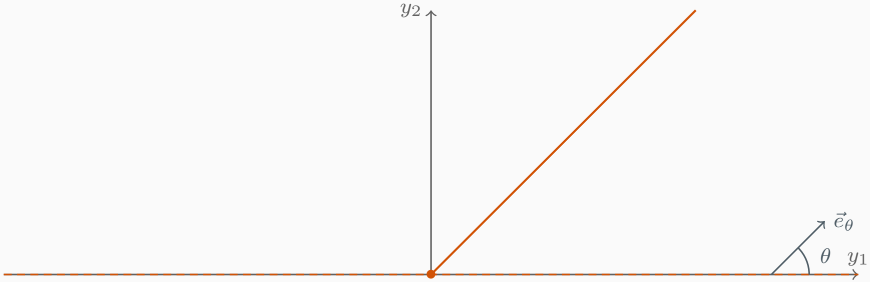
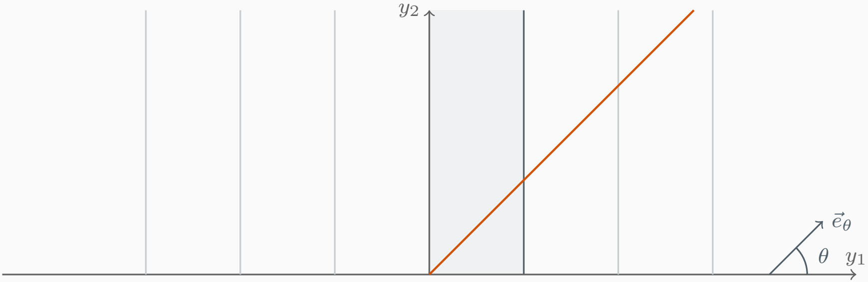
$$u_\theta(0) = 1 \longrightarrow U_\theta = \varphi, \quad \text{on } \mathbb{R}\times\{0\}, \quad \text{with } \varphi(0) = 1$$



Extension to a periodic half-guide problem

Theorem (periodic boundary data)

If $\varphi(y_1 + 1) = \varphi(y_1)$, then $U_\theta(y_1 + 1, y_2) = U_\theta(y_1, y_2)$



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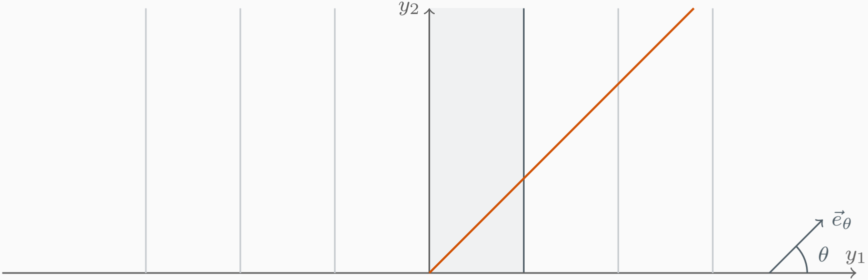
$$u_\theta(0) = 1.$$

$\longrightarrow$

$$\begin{aligned} -D_\theta\left(\mu_p\, D_\theta U_\theta\right) - \rho_p\, \omega^2\, U_\theta &= 0, \quad (0,1)\times\mathbb{R}_+^* \\ U_\theta &= \varphi, \quad (0,1)\times\{0\} \\ U_\theta(y_1+1, y_2) &= U_\theta(y_1, y_2) \end{aligned}$$



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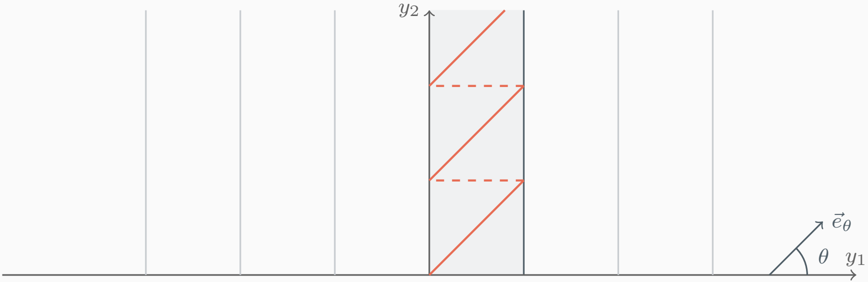

The periodic half-guide problem

Periodic half-guide problem with absorption

$$u_\theta(x) = U_\theta(x\,\vec{e}_\theta) \quad \left| \begin{array}{rl} -D_\theta\left(\mu_p\, D_\theta U_\theta\right) - \rho_p\,\omega^2\, U_\theta = 0, & \mathcal{B}_0 := (0,1)\times\mathbb{R}_+^* \\ U_\theta = \varphi, & (0,1)\times\{0\} \\ U_\theta|_{y_1=0} = U_\theta|_{y_1=1} \quad \mu_p D_\theta U_\theta|_{y_1=0} = \mu_p D_\theta U_\theta|_{y_1=1} & \end{array} \right. \qquad (\mathscr{P}_{\text{per}})$$

- **Pros** Periodic coefficients
- **Cons** Nonelliptic principal part

The periodic half-guide problem

Periodic half-guide problem **with absorption**

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Functional framework

$$H^1_\theta(\mathcal{B}_0) = \left\{ V \in L^2(\mathcal{B}_0) \;/\; D_\theta V \in L^2(\mathcal{B}_0) \right\}$$

$$H^1_{\text{per},\theta}(\mathcal{B}_0) = \left\{ V \in H^1_\theta(\mathcal{B}_0) \;/\; V|_{y_1=0} = V|_{y_1=1} \right\}$$

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$$H^1_{\text{per},\theta}(\mathcal{B}_0) = \left\{V \in H^1_\theta(\mathcal{B}_0) \;/\; V|_{y_1=0} = V|_{y_1=1}\right\}$$

Theorem (Well posedness)

For all data $\varphi \in L^2(0,1)$, $(\mathscr{P}_{\text{per}})$ admits a unique solution $U_\theta(\varphi) \in H^1_{\text{per},\theta}(\mathcal{B}_0)$.

The periodic half-guide problem

Periodic half-guide problem **with absorption**

$$u_\theta(x) = U_\theta(x\,\vec{e}_\theta) \quad \left| \begin{array}{rl} -D_\theta\,(\mu_p\,D_\theta U_\theta) - \rho_p\,\omega^2\,U_\theta = 0, & \mathcal{B}_0 := (0,1)\times\mathbb{R}^*_+ \\ U_\theta = \varphi, & (0,1)\times\{0\} \\ U_\theta|_{y_1=0} = U_\theta|_{y_1=1} \quad \mu_p D_\theta U_\theta|_{y_1=0} = \mu_p D_\theta U_\theta|_{y_1=1} & \end{array} \right. \qquad (\mathscr{P}_{\text{per}})$$

Theorem (Regularity in all directions)

Assume that $\partial_{y_1}\mu_p,\ \partial_{y_1}\rho_p \in L^\infty(0,1)^2$ and $\varphi \in H^1(0,1)$. Then $U_\theta(\varphi) \in H^1(\mathcal{B}_0)$.

The periodic half-guide problem

Periodic half-guide problem **with absorption**

$$u_\theta(x) = U_\theta(x\,\vec{e}_\theta) \quad \left|\begin{array}{rl} -D_\theta\left(\mu_p\, D_\theta U_\theta\right) - \rho_p\,\omega^2\, U_\theta = 0, & \mathcal{B}_0 := (0,1)\times\mathbb{R}_+^* \\ U_\theta = \varphi, & (0,1)\times\{0\} \\ U_\theta|_{y_1=0} = U_\theta|_{y_1=1} \quad \mu_p D_\theta U_\theta|_{y_1=0} = \mu_p D_\theta U_\theta|_{y_1=1} & \end{array}\right. \qquad (\mathscr{P}_{per})$$

Theorem (Regularity in all directions)

Assume that $\partial_{y_1}\mu_p,\ \partial_{y_1}\rho_p \in L^\infty(0,1)^2$ and $\varphi \in H^1(0,1)$. Then $U_\theta(\varphi) \in H^1(\mathcal{B}_0)$.

How can one solve $(\mathscr{P}_{per})$ numerically?

Numerical resolution of elliptic **periodic** PDE in unbounded domains

Fliss, *Joly*, *Li*, 2006   *Fliss*, 2009   *Fliss*, *Joly*, *Lescarret*, 2020

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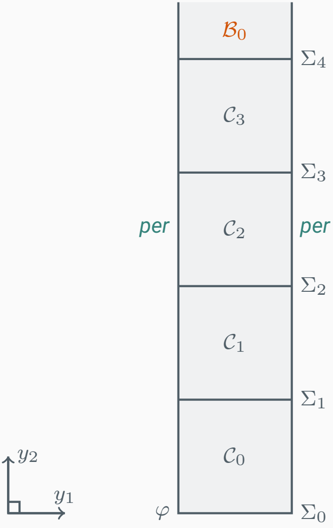
Structure of the solution

As the solution of a **periodic** half-guide problem, $U_\theta(\varphi)$ has a certain structure.



$$\Sigma_\ell \equiv \Sigma \quad \mathcal{C}_n \equiv \mathcal{C}$$

$$-D_\theta\,(\mu_p\, D_\theta U_\theta) - \rho_p\,\omega^2\, U_\theta = 0, \qquad \mathcal{B}_0$$

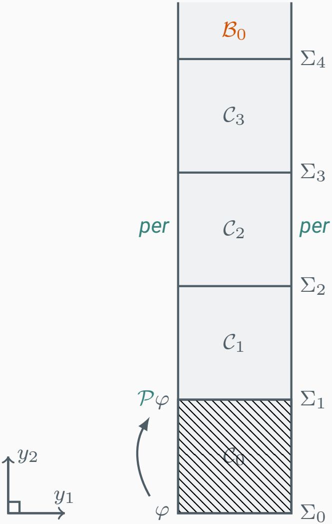
$$U_\theta = \varphi, \qquad \Sigma_0$$

$\oplus$ Periodicity conditions

- Which PDE does $U_\theta(\varphi)(\cdot, \cdot + 1)$ satisfy?

Structure of the solution

As the solution of a **periodic** half-guide problem, $U_\theta(\varphi)$ has a certain structure.



$$\Sigma_\ell \equiv \Sigma \quad \mathcal{C}_n \equiv \mathcal{C}$$

$$\mathcal{P}\varphi = U_\theta(\varphi)|_{\Sigma_1}$$

$$-D_\theta\left(\mu_p\, D_\theta U_\theta\right) - \rho_p\,\omega^2\, U_\theta = 0, \qquad \mathcal{B}_0 \setminus \mathcal{C}_0$$

$$U_\theta = \mathcal{P}\varphi, \qquad \Sigma_1$$

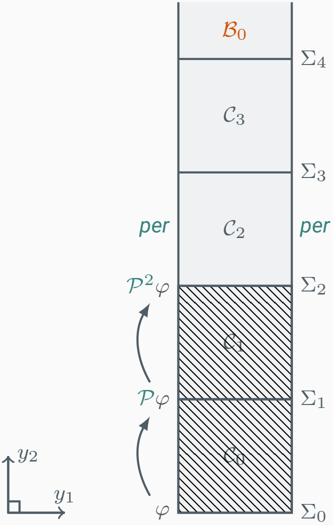
$\oplus$ Periodicity conditions

$$U_\theta(\varphi)(\cdot, \cdot + 1) = U_\theta(\mathcal{P}\varphi)$$

- $U(\varphi)(\cdot, \cdot + 1)$ satisfies the same PDE as $U(\varphi)$
- But with a different Dirichlet boundary data

Structure of the solution

As the solution of a **periodic** half-guide problem, $U_\theta(\varphi)$ has a certain structure.



$$\Sigma_\ell \equiv \Sigma \qquad \mathcal{C}_n \equiv \mathcal{C}$$

$$\mathcal{P}^2\varphi = U_\theta(\varphi)|_{\Sigma_2}$$

$$-D_\theta\ (\mu_p\ D_\theta U_\theta) - \rho_p\ \omega^2\ U_\theta = 0, \qquad \mathcal{B}_0 \setminus \mathcal{C}_0 \cup \mathcal{C}_1$$

$$U_\theta = \mathcal{P}^2\varphi, \quad \Sigma_2$$

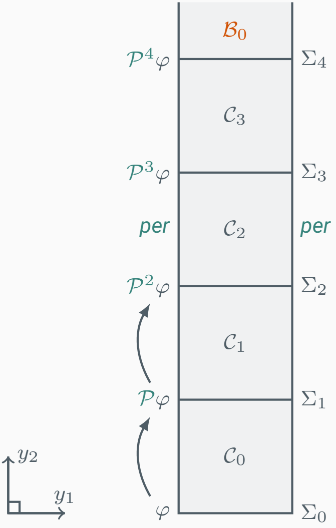
$\oplus$ Periodicity conditions

$$U_\theta(\varphi)(\cdot, \cdot + 2) = U_\theta(\mathcal{P}\varphi)(\cdot, \cdot + 1) = U_\theta(\mathcal{P}^2\varphi)$$

- $U(\varphi)(\cdot, \cdot + 2)$ satisfies the same PDE as $U(\varphi)$
- But with a different Dirichlet boundary data

Structure of the solution

As the solution of a **periodic** half-guide problem, $U_\theta(\varphi)$ has a certain structure.



$$\Sigma_\ell \equiv \Sigma \quad \mathcal{C}_n \equiv \mathcal{C}$$

$$\boxed{\mathcal{P}\varphi = U_\theta(\varphi)|_{\Sigma_1}}$$

By induction,

Theorem (Structure of the solution)

$$\forall\, n > 0, \quad U_\theta(\varphi)(\cdot, \cdot + n) = U_\theta(\mathcal{P}^n\varphi)$$

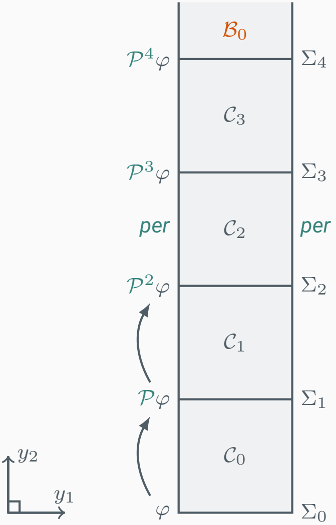
$\mathcal{P}$ is called the propagation operator

Theorem (Properties of $\mathcal{P}$ when $\mathrm{Im}\,(\omega) > 0$)

- $\mathcal{P}$ is injective and uniquely defined
- $\mathcal{P}$ has a spectral radius $\rho(\mathcal{P}) < 1$

Structure of the solution

As the solution of a **periodic** half-guide problem, $U_\theta(\varphi)$ has a certain structure.



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By induction,

Theorem (Structure of the solution)

$$\forall\, n > 0, \quad U_\theta(\varphi)(\cdot, \cdot + n) = U_\theta(\mathcal{P}^n\varphi)$$

$\mathcal{P}$ is called the propagation operator

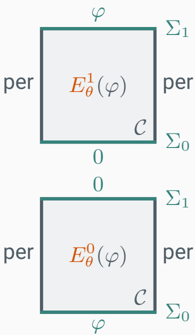
Theorem (Non-compactness of $\mathcal{P}$)

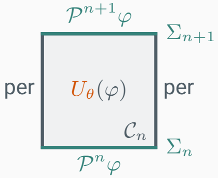
- $\mathcal{P}$ has continuous spectrum
- If $\cot\theta$ is irrational, then $\sigma(\mathcal{P})$ is a circle

Solutions of local cell problems

Given a data φ, compute the solutions $E_\theta^0(\varphi)$ and $E_\theta^1(\varphi)$ of local cell problems

$$-D_\theta\ \left(\mu_p\ D_\theta E_\theta^\ell\right) - \rho_p\ \omega^2\ E_\theta^\ell = 0, \quad \mathcal{C}$$





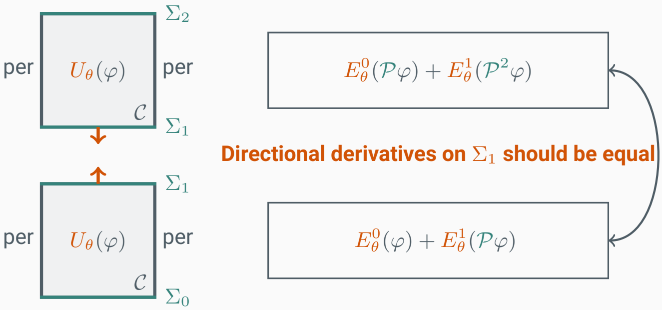
$\Downarrow$ Linearity

$$U_\theta(\varphi)(\cdot, \cdot + n) = E_\theta^0(\mathcal{P}^n\varphi) + E_\theta^1(\mathcal{P}^{n+1}\varphi)$$

Solutions of local cell problems

Given a data φ, compute the solutions $E_\theta^0(\varphi)$ and $E_\theta^1(\varphi)$ of local cell problems

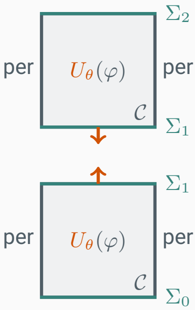
$$-D_\theta \, (\mu_p \, D_\theta E_\theta^\ell) - \rho_p \, \omega^2 \, E_\theta^\ell = 0, \quad \mathcal{C}$$



Local Dirichlet-to-Neumann operators

Given a data φ, compute the solutions $E_\theta^0(\varphi)$ and $E_\theta^1(\varphi)$ of local cell problems

$$-D_\theta\ \left(\mu_p\ D_\theta E_\theta^\ell\right) - \rho_p\ \omega^2\ E_\theta^\ell = 0, \quad \mathcal{C}$$

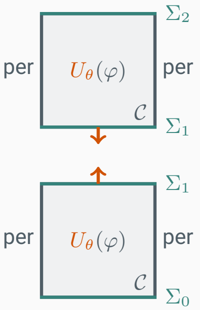


$$D_\theta E_\theta^0(\mathcal{P}\varphi)\big|_{\Sigma_0} + D_\theta E_\theta^1(\mathcal{P}^2\varphi)\big|_{\Sigma_0}$$
$$\parallel$$
$$D_\theta E_\theta^0(\varphi)\big|_{\Sigma_1} + D_\theta E_\theta^1(\mathcal{P}\varphi)\big|_{\Sigma_1}$$

Local Dirichlet-to-Neumann operators

Given a data φ, compute the local DtN operators $\mathcal{T}^{00}, \mathcal{T}^{01}, \mathcal{T}^{10}, \mathcal{T}^{11} \in \mathcal{L}(L^2(\Sigma))$

$$\mathcal{T}^{\ell j}\varphi = (-1)^{j+1}\ D_\theta E^\ell_\theta(\varphi)\Big|_{\Sigma_j}$$



$$-\mathcal{T}^{00}\mathcal{P}\varphi - \mathcal{T}^{10}\mathcal{P}^2\varphi = \mathcal{T}^{01}\varphi + \mathcal{T}^{11}\mathcal{P}\varphi$$

Construction of the solution

Local Dirichlet-to-Neumann operators

Given a data φ, compute the local DtN operators $\mathcal{T}^{00}, \mathcal{T}^{01}, \mathcal{T}^{10}, \mathcal{T}^{11} \in \mathcal{L}(L^2(\Sigma))$

$$\mathcal{T}^{\ell j}\varphi = (-1)^{j+1}\; D_\theta E_\theta^\ell(\varphi)\Big|_{\Sigma_j}$$

Theorem (Characterization of $\mathcal{P}$ **when** $\operatorname{Im}(\omega) > 0$)

The operator $\mathcal{P}$ is the unique solution of the stationary Riccati equation

Find $\mathcal{P} \in \mathcal{L}(L^2(\Sigma))$ *such that* $\rho(\mathcal{P}) < 1$ *and*

$$\mathcal{T}^{10}\,\mathcal{P}^2 + (\mathcal{T}^{00} + \mathcal{T}^{11})\,\mathcal{P} + \mathcal{T}^{01} = 0. \qquad (\mathscr{R})$$

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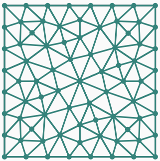
Algorithm and numerical results

Solve the periodic waveguide problem

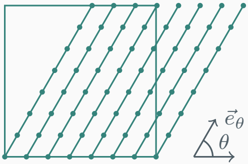
1. Compute the solutions $E_\theta^0(\varphi)$ and $E_\theta^1(\varphi)$ of local cell problems

2D finite elements



- Solve the local cell problems on an unstructured 2D mesh

1D finite elements along $\vec{e}_\theta$



- Solve 1D quasiperiodic cell problems along $\vec{e}_\theta$
- Concatenate the 1D solutions

Solve the periodic waveguide problem

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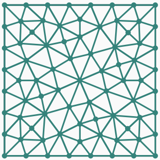

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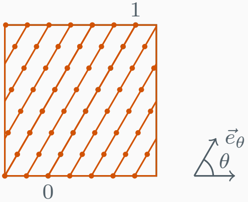

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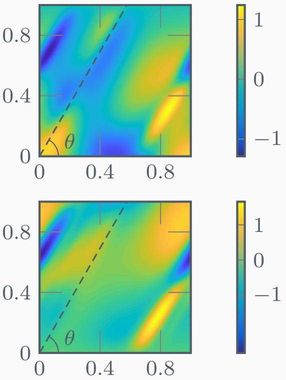

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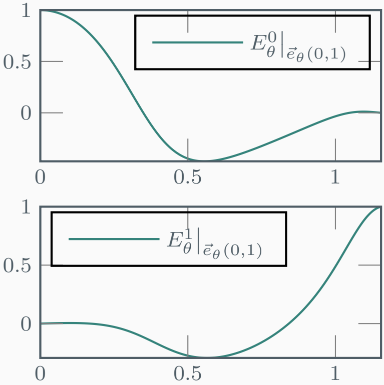


Algorithm and numerical results

Solve the periodic waveguide problem

1. Compute the solutions $E_\theta^0(\varphi)$ and $E_\theta^1(\varphi)$ of local cell problems
2. Compute the local DtN operators $\mathcal{T}^{00}, \mathcal{T}^{01}, \mathcal{T}^{10}, \mathcal{T}^{11}$

2D finite elements

- Weak evaluation

$$\int_\Sigma \mathcal{T}^{\ell j} \varphi \, \overline{\psi} = \int_\mathcal{C} \mu_p \, D_\theta E_\theta^\ell(\varphi) \, D_\theta \overline{E_\theta^j(\psi)} - \rho_p \, \omega^2 \, E_\theta^\ell(\varphi) \, \overline{E_\theta^j(\psi)}$$

1D finite elements along $\vec{e}_\theta$

- Interpolation

Solve the periodic waveguide problem

1. Compute the solutions $E_\theta^0(\varphi)$ and $E_\theta^1(\varphi)$ of local cell problems
2. Compute the local DtN operators $\mathcal{T}^{00}, \mathcal{T}^{01}, \mathcal{T}^{10}, \mathcal{T}^{11}$

2D finite elements

- Weak evaluation

$$\int_\Sigma \mathcal{T}^{\ell j}\varphi\,\overline{\psi} = \int_\mathcal{C} \mu_p\, D_\theta E_\theta^\ell(\varphi)\, D_\theta \overline{E_\theta^j(\psi)} - \rho_p\, \omega^2\, E_\theta^\ell(\varphi)\, \overline{E_\theta^j(\psi)}$$

1D finite elements along $\vec{e}_\theta$

- Interpolation

$N \times N$ matrices
where N is the number of DOFs associated to Σ

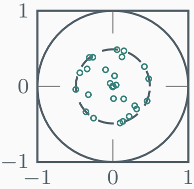
Algorithm and numerical results

Solve the periodic waveguide problem

1. Compute the solutions $E^0_\theta(\varphi)$ and $E^1_\theta(\varphi)$ of local cell problems
2. Compute the local DtN operators $\mathcal{T}^{00}, \mathcal{T}^{01}, \mathcal{T}^{10}, \mathcal{T}^{11}$
3. Determine the unique solution $\mathcal{P}$ with a spectral radius $\rho(\mathcal{P}) < 1$ of the equation

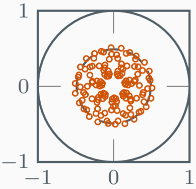
$$\mathcal{T}^{10}\,\mathcal{P}^2 + (\mathcal{T}^{00} + \mathcal{T}^{11})\,\mathcal{P} + \mathcal{T}^{01} = 0$$

2D finite elements



Spectral method

1D finite elements along $\vec{e}_\theta$



- $N = 128$
- $\omega = 5 + 0.5\imath$
- $\theta = \pi/3$

Algorithm and numerical results

Solve the periodic waveguide problem

1. Compute the solutions $E_\theta^0(\varphi)$ and $E_\theta^1(\varphi)$ of local cell problems
2. Compute the local DtN operators $\mathcal{T}^{00}, \mathcal{T}^{01}, \mathcal{T}^{10}, \mathcal{T}^{11}$
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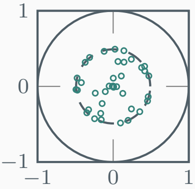
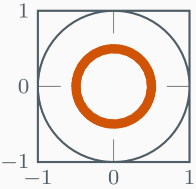
2D finite elements



Spectral method

1D finite elements along $\vec{e}_\theta$



• $N = 512$ • $\omega = 5 + 0.5\imath$ • $\theta = \pi/3$

Algorithm and numerical results

Solve the periodic waveguide problem

1. Compute the solutions $E_\theta^0(\varphi)$ and $E_\theta^1(\varphi)$ of local cell problems
2. Compute the local DtN operators $\mathcal{T}^{00}, \mathcal{T}^{01}, \mathcal{T}^{10}, \mathcal{T}^{11}$
3. Determine the unique solution $\mathcal{P}$ with a spectral radius $\rho(\mathcal{P}) < 1$ of the equation
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4. Construct the solution $U_\theta(\varphi)$ cell by cell

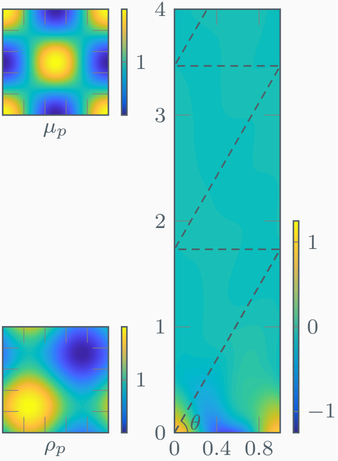
$$U_\theta(\varphi)(\cdot, \cdot + n)|_{\mathcal{C}} = E_\theta^0(\mathcal{P}^n \varphi) + E_\theta^1(\mathcal{P}^{n+1} \varphi)$$

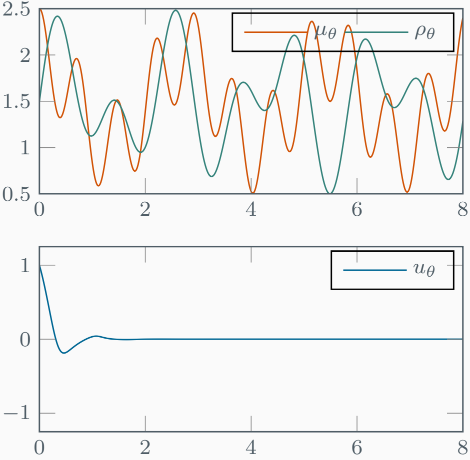
Solve the quasiperiodic half-line problem

Compute $u_\theta(x) = U_\theta(x\, \vec{e}_\theta)$

Test case for the locally perturbed quasiperiodic problem

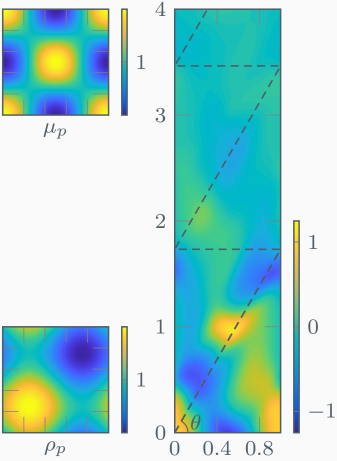
- $\omega = 5 + 3\imath$
- $\theta = \pi/3$

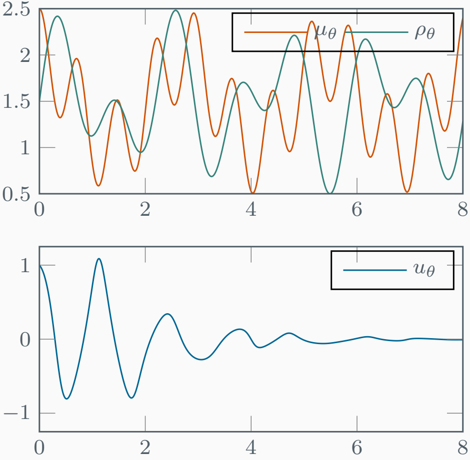




Test case for the locally perturbed quasiperiodic problem

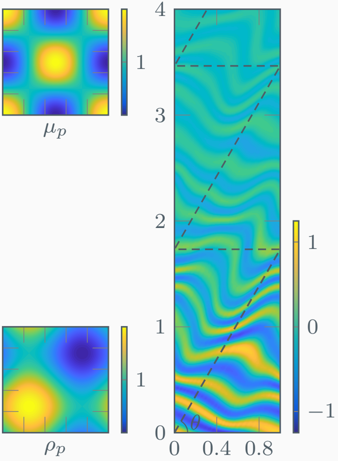
- $\omega = 5 + 0.5\imath$
- $\theta = \pi/3$

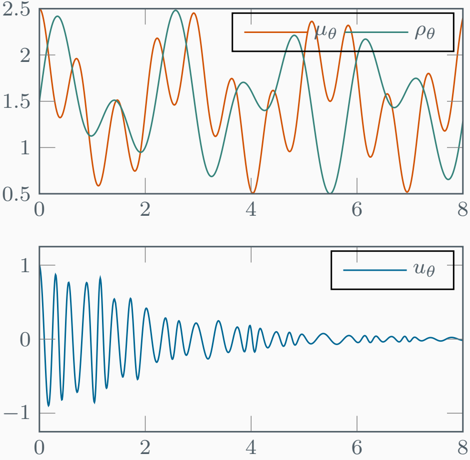




Test case for the locally perturbed quasiperiodic problem

- $\omega = 20 + 0.5\imath$
- $\theta = \pi/3$





Algorithm and numerical results

Solve the periodic waveguide problem

1. Compute the solutions $E^0_\theta(\varphi)$ and $E^1_\theta(\varphi)$ of local cell problems
2. Compute the local DtN operators $\mathcal{T}^{00}, \mathcal{T}^{01}, \mathcal{T}^{10}, \mathcal{T}^{11}$
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4. Construct the solution $U_\theta(\varphi)$ cell by cell

Solve the quasiperiodic half-line problem

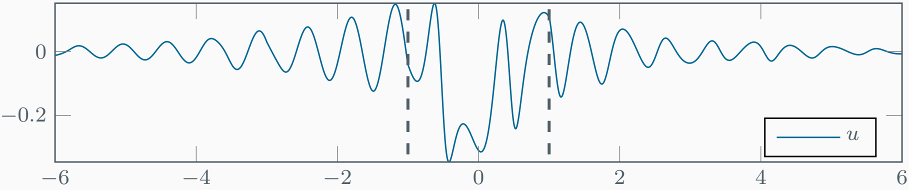
Compute $u_\theta(x) = U_\theta(x\, \vec{e}_\theta)$

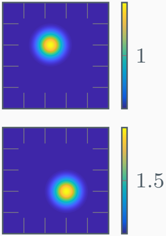
Solve the locally perturbed quasiperiodic problem

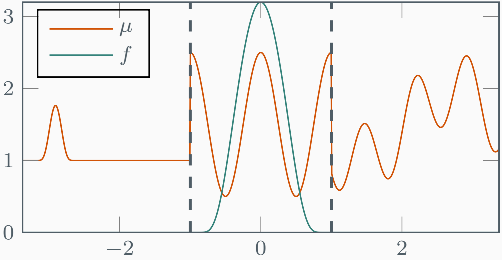
Construct the global solution u of the locally perturbed quasiperiodic problem $(\mathscr{P})$

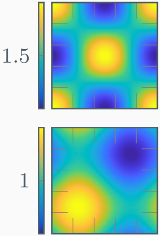
Test case for the locally perturbed quasiperiodic problem

- $\omega = 10 + 0.5\imath$
- $\theta^+ = \pi/3$
- $\theta^- = \pi/6$





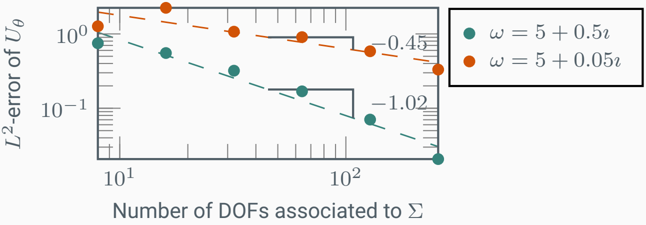




Passing the absorption to the limit

Observation

The numerical approximation deteriorates as $\mathrm{Im}\,(\omega)$ tends to 0.



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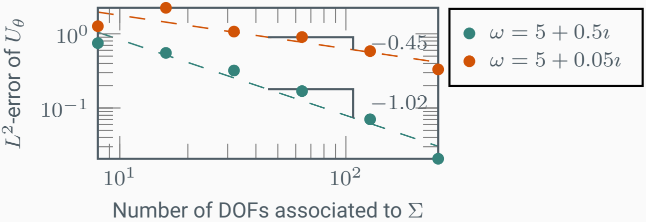

Theorem (Ill-posedness of the local cell problems without absorption)

If μ_p and ρ_p are not constant, and if $\cot\theta$ is irrational, there exists $\omega_{\min} \in \mathbb{R}$ such that for

$$\omega \in (\omega_{\min}, +\infty),$$

the local cell problems with Dirichlet boundary conditions are ill-posed.

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Conclusion

Summary

Resolution of the **Helmholtz equation** in 1D locally perturbed **quasiperiodic** media

- Extend the quasiperiodic PDE to a periodic PDE through the **cut approach**

The case without absorption

- Non-uniqueness for the Riccati equation $\longleftarrow$ **Also true in the periodic case**
  - Additional condition to fully characterize $\mathcal{P}$
- Ill-posedness of Dirichlet-type local cell problems $\longleftarrow$ **Specific to the quasiperiodic case**
  - Solve Robin-type local cell problems instead

Conclusion

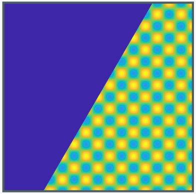
Summary

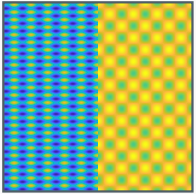
Resolution of the **Helmholtz equation** in 1D locally perturbed **quasiperiodic** media

- Extend the quasiperiodic PDE to a periodic PDE through the **cut approach**

The multidimensional case

- Extension to quasiperiodic functions of several variables
- Application to transmission problems





Conclusion

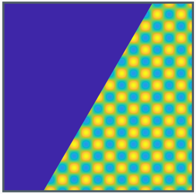
Summary

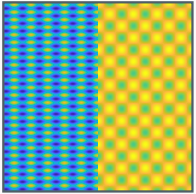
Resolution of the **Helmholtz equation** in 1D locally perturbed **quasiperiodic** media

- Extend the quasiperiodic PDE to a periodic PDE through the **cut approach**

The multidimensional case

- Extension to quasiperiodic functions of several variables
- Application to transmission problems





Thank you for your attention!