DE LA RECHERCHE À L'INDUSTRIE

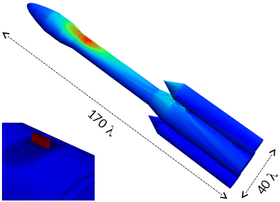
Décomposition de domaine sur des formulations intégrales surfaciques en électromagnétisme

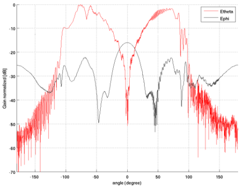
Francis Collino Justine Labat Agnès Pujols Muriel Sesques | CEA-CESTA

SMAI 2021

Context and motivations

- Context : Solution of electromagnetic wave scattering problem by a complex object using a boundary element method
- Motivation : Accurate evaluation of radar cross-sections
  - Large-scale objects (in comparison with the wavelength)
  - Multi-scale phenomena





Example : Monopole antenna in the presence of a dielectric object on a launcher (ISAE Workshop, 2016)

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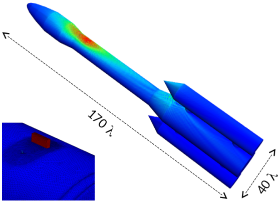
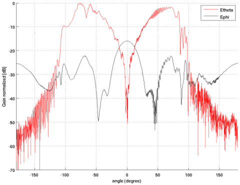


- Motivation : Accurate evaluation of radar cross-sections
  - Large-scale objects (in comparison with the wavelength)
  - Multi-scale phenomena

- Difficulties : High computational costs
  - Number of degrees of freedom : 1 650 875
    → Approximately 100 000 000 in volume

Solver	Number of CPU	Factorization (in CPU·H)
LL^t (prediction)	15 000	151 435
Block low-rank	800	2 541
H-matrix	200	1 043

Hackbush (1999), Bebendorf (2008), . . .



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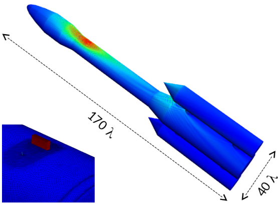
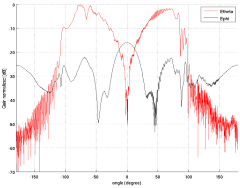


- Difficulties : High computational costs

- Objectives : Development of a domain decomposition method
  - Robust simulations for a wide range of frequencies
    → Solve very large problems (more than 10 000 000 of unknowns)
  - Geometry-adaptative strategies to handle multi-scale structures
    → Allow non-conformal meshes
  - Simulation-based engineering using High Performance Computing



Example : Monopole antenna in the presence of a dielectric object on a launcher (ISAE Workshop, 2016)

- Adaptation of volumical discontinuous Galerkin methods to surface problems

  Peng, Hiptmair and Shao (2016), Messai and Pernet (2020)

	Volume	Surface	Comments
Locality	✓	✗	Iterative solution is inevitable
Restrictions	✓	✗	Restrictions of distributions are not easy to define
Traces	✓	✗	No trace theorem in $\mathsf{H}_t^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$

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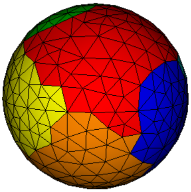
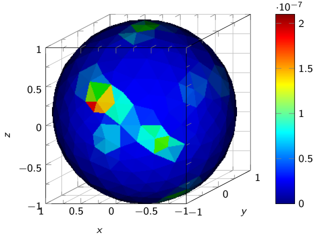
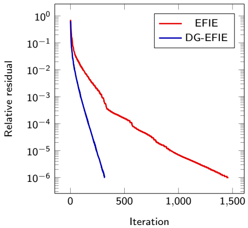
- Why not to directly use an iterative solver on BEM ?
  - Main motivation : non-conformal meshes
  - EFIE simulation preconditioned using domain partitioning on a low-frequency case







Steinbach and Wendland (1998), Christiansen and Nédélec (2000), Antoine, Bendali and Darbas (2005), Andriulli *et al.* (2008)

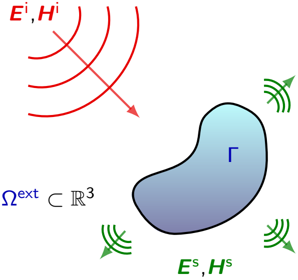
Outline

1. Discontinuous Galerkin-based surface domain decomposition method
   - Model problem
   - Discontinuous formulation
   - Discrete formulation
   - Iterative procedure

2. Numerical results
   - Accuracy of solutions
   - Convergence of the iterative solver
   - Computational costs

3. Conclusion and perspectives

Outline



- Using a time-harmonic dependence in $\exp(+\mathrm{i}\omega t)$

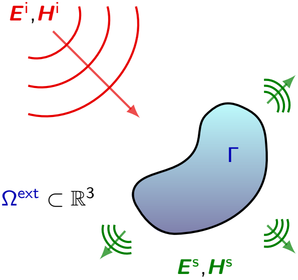
$$
\begin{aligned}
&\nabla \times \boldsymbol{E}^{\mathrm{s}} + \mathrm{i}\kappa Z_0^{-1} \boldsymbol{H}^{\mathrm{s}} = 0 && \text{in } \Omega^{\mathrm{ext}} \\
&\nabla \times \boldsymbol{H}^{\mathrm{s}} - \mathrm{i}\kappa Z_0 \boldsymbol{E}^{\mathrm{s}} = 0 && \text{in } \Omega^{\mathrm{ext}} \\
&\boldsymbol{n} \times \boldsymbol{E}^{\mathrm{s}} = -\boldsymbol{n} \times \boldsymbol{E}^{\mathrm{i}} && \text{on } \Gamma \\
&\lim_{|\boldsymbol{x}| \to \infty} |\boldsymbol{x}| \left(Z_0 \boldsymbol{H}^{\mathrm{s}} \times \hat{x} - \boldsymbol{E}^{\mathrm{s}} \right) = 0 \quad \text{unif. in } \hat{x} = \frac{\boldsymbol{x}}{|\boldsymbol{x}|}
\end{aligned}
$$

κ : wave-number, Z_0 : impedance coefficient in vacuum

- Ω^{ext} : Exterior domain
- Γ : Scattering surface
- $\boldsymbol{E}^{\mathrm{i}}, \boldsymbol{H}^{\mathrm{i}}$: Incident fields
- $\boldsymbol{E}^{\mathrm{s}}, \boldsymbol{H}^{\mathrm{s}}$: Scattered fields
- $\boldsymbol{E}, \boldsymbol{H}$: Total fields

$$\boldsymbol{E} = \boldsymbol{E}^{\mathrm{i}} + \boldsymbol{E}^{\mathrm{s}} \qquad \boldsymbol{H} = \boldsymbol{H}^{\mathrm{i}} + \boldsymbol{H}^{\mathrm{s}}$$

The exterior time-harmonic Maxwell problem



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\end{aligned}$$

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- Using the Stratton-Chu formulas for $\boldsymbol{x} \in \Omega^{\mathrm{ext}}$

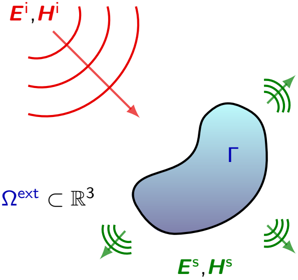
$$\boldsymbol{E}^{\mathrm{s}}(\boldsymbol{x}) = -\mathrm{i}\kappa \mathcal{T} \boldsymbol{J}(\boldsymbol{x}) \quad \text{and} \quad \boldsymbol{H}^{\mathrm{s}}(\boldsymbol{x}) = \frac{1}{Z_0} \mathcal{K} \boldsymbol{J}(\boldsymbol{x})$$

$\boldsymbol{J} = Z_0(\boldsymbol{n} \times \boldsymbol{H})$: total surface electric current

$$\mathcal{T}\boldsymbol{J} = \frac{1}{\kappa^2} \nabla \left(\mathcal{S} \operatorname{div}_\Gamma \boldsymbol{J} \right) + \mathcal{S}\boldsymbol{J}$$

$$\mathcal{K}\boldsymbol{J} = \nabla \times \mathcal{S}\boldsymbol{J} \qquad \tilde{\mathcal{S}}\lambda(\boldsymbol{x}) = \int_\Gamma G(\boldsymbol{x}, \boldsymbol{y}) \lambda(\boldsymbol{y}) \, \mathrm{d}s_{\boldsymbol{y}}$$

G : out-going Green function of the Helmholtz equation



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- E^i, H^i : Incident fields
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$$\boldsymbol{E} = \boldsymbol{E}^i + \boldsymbol{E}^s \qquad \boldsymbol{H} = \boldsymbol{H}^i + \boldsymbol{H}^s$$

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- Jump relations give boundary integral equations

$$\begin{aligned}
&\textbf{(EFIE)} \qquad \mathrm{i}\kappa T\boldsymbol{J} = \boldsymbol{n} \times (\boldsymbol{E}^i \times \boldsymbol{n}) && \text{on } \Gamma\\
&\textbf{(MFIE)} \qquad \frac{1}{2}\boldsymbol{J} - K\boldsymbol{J} = Z_0(\boldsymbol{n} \times \boldsymbol{H}^i) && \text{on } \Gamma
\end{aligned}$$

(EFIE) $\quad \mathrm{i}\kappa T\boldsymbol{J} = \boldsymbol{n} \times (\boldsymbol{E}^{\mathrm{i}} \times \boldsymbol{n}) \quad \text{on } \Gamma$

(MFIE) $\quad \frac{1}{2}\boldsymbol{J} - K\boldsymbol{J} = Z_0(\boldsymbol{n} \times \boldsymbol{H}^{\mathrm{i}}) \quad \text{on } \Gamma$

$$T : \mathrm{H}_{\mathrm{t}}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \longrightarrow \mathrm{H}_{\mathrm{t}}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$$

$$T\boldsymbol{J} = \frac{1}{\kappa^2}\nabla_\Gamma\left(S\mathrm{div}_\Gamma \boldsymbol{J}\right) + S\boldsymbol{J}$$

$$K : \mathrm{H}_{\mathrm{t}}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \longrightarrow \mathrm{H}_{\mathrm{t}}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$$

$$K\boldsymbol{J} = \mathrm{p.v.}\left[\boldsymbol{n} \times (\nabla_\Gamma \times S\boldsymbol{J})\right]$$

(**EFIE**) $\quad \mathrm{i}\kappa T\boldsymbol{J} = \boldsymbol{n} \times (\boldsymbol{E}^{\mathrm{i}} \times \boldsymbol{n})$ on Γ

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- $\gamma_{nm} = \Gamma_n \cap \Gamma_m$
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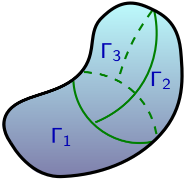

- $\boldsymbol{\tau}_{nm}$: exterior normal vector to γ_{nm}
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$$\mathrm{i}\kappa \sum_{m=1}^{N} T_{nm} \boldsymbol{J}_m = \boldsymbol{n} \times (\boldsymbol{E}^{\mathrm{i}} \times \boldsymbol{n}) \quad \text{on } \Gamma_n$$

$$T_{nm} : \widetilde{\mathrm{H}}_{\mathrm{t}}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_m}, \Gamma_m) \longrightarrow \widetilde{\mathrm{H}}_{\mathrm{t}}^{-\frac{1}{2}}(\mathrm{curl}_{\Gamma_n}, \Gamma_n)$$

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1 Restriction to each subdomain

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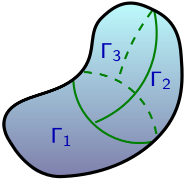

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2 Obtention of variational formulations

- Multiplication by a test-function
- Integration on partial surfaces
- Integration by parts (EFIE only)

$$\langle \nabla_{\Gamma}\left(S_{nm}\mathrm{div}_{\Gamma}\boldsymbol{J}_m\right), \boldsymbol{v}_n\rangle_{\Gamma_n} = -\langle S_{nm}\mathrm{div}_{\Gamma}\boldsymbol{J}_m, \mathrm{div}_{\Gamma}\boldsymbol{v}_n\rangle_{\Gamma_n} + \langle S_{nm}\mathrm{div}_{\Gamma}\boldsymbol{J}_m, \boldsymbol{v}_n \cdot \boldsymbol{\tau}_n\rangle_{\gamma_n}$$

- which requires more regularity ($\varepsilon > 0$)

$$\mathrm{div}_{\Gamma}\boldsymbol{J}_m \in H^{-\frac{1}{2}+\varepsilon}(\Gamma_m) \text{ and } \boldsymbol{v}_n \in \mathrm{H}_t^{\frac{1}{2}-\varepsilon}(\mathrm{div}_{\Gamma}, \Gamma_n)$$

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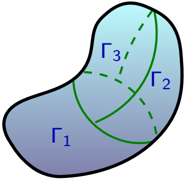

Discontinuous boundary integral equations : general principle

$$\mathrm{i}\kappa \sum_{m=1}^{N} T_{nm} \boldsymbol{J}_m = \boldsymbol{n} \times (\boldsymbol{E}^{\mathrm{i}} \times \boldsymbol{n}) \quad \text{on } \Gamma_n$$

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2 Obtention of variational formulations

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- Integration on partial surfaces
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3 Summation over all the subdomains

$$\sum_{n=1}^{N} \langle S_{nm} \mathrm{div}_{\Gamma} \boldsymbol{J}_m, \boldsymbol{v}_n \cdot \boldsymbol{\tau}_n \rangle_{\gamma_n} = \sum_{\gamma_{nm}} \langle S_{nm} \mathrm{div}_{\Gamma} \boldsymbol{J}_m, [\boldsymbol{v}]_{\gamma_{nm}} \rangle_{\gamma_{nm}}$$

$[\boldsymbol{v}]_{\gamma_{nm}} = \boldsymbol{\tau}_{nm} \cdot \boldsymbol{v}_m + \boldsymbol{\tau}_{mn} \cdot \boldsymbol{v}_n$: jump across γ_{nm}

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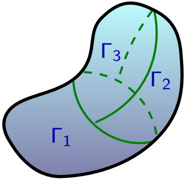

- $\gamma_{nm} = \Gamma_n \cap \Gamma_m$
- $\gamma_n = \partial \Gamma_n$
- $\boldsymbol{\tau}_{nm}$: exterior normal vector to γ_{nm}
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$$\text{Find } \boldsymbol{J} \in \bigoplus_{m=1}^{N} \mathrm{H}_{\mathrm{t}}^{-\frac{1}{2}+\varepsilon}(\mathrm{div}_{\Gamma_m}, \Gamma_m) \text{ with } \varepsilon > 0 \text{ such that}$$

$$a_\Gamma(\boldsymbol{J}, \boldsymbol{v}) + a_\gamma^{\pm}(\boldsymbol{J}, \boldsymbol{v}) + p_\gamma^{*}(\boldsymbol{J}, \boldsymbol{v}) = \ell(\boldsymbol{v}) \qquad \text{for any } \boldsymbol{v} \in \bigoplus_{n=1}^{N} \mathrm{H}_{\mathrm{t}}^{\frac{1}{2}-\varepsilon}(\mathrm{div}_{\Gamma_n}, \Gamma_n)$$

$$a_\Gamma(\boldsymbol{J}, \boldsymbol{v}) = \sum_{n=1}^{N} \sum_{m=1}^{N} \left\{ \frac{1}{\mathrm{i}\kappa} \left\langle S_{nm} \mathrm{div}_{\Gamma_m} \boldsymbol{J}_m, \mathrm{div}_{\Gamma_n} \boldsymbol{v}_n \right\rangle_{\Gamma_n} + \mathrm{i}\kappa \left\langle S_{nm} \boldsymbol{J}_m, \boldsymbol{v}_n \right\rangle_{\Gamma_n} \right\} \qquad \text{(continuous EFIE)}$$

$$a_\gamma^{\pm}(\boldsymbol{J}, \boldsymbol{v}) = -\frac{1}{\mathrm{i}\kappa} \sum_{m=1}^{N} \sum_{\gamma_{nm}} \left\langle S_{nm} \mathrm{div}_{\Gamma_m} \boldsymbol{J}_m, [\boldsymbol{v}]_{\gamma_{nm}} \right\rangle_{\gamma_{nm}} + \widetilde{a}_\gamma^{\pm}(\boldsymbol{J}, \boldsymbol{v}) \qquad \text{(symmetrization)}$$

$p_\gamma^{*}(\boldsymbol{J}, \boldsymbol{v})$ is function of $[\boldsymbol{J}]_{\gamma_{nm}}$ and $[\boldsymbol{v}]_{\gamma_{nm}}$ (penalization)

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$$a_\gamma^{\pm}(\boldsymbol{J}, \boldsymbol{v}) = -\frac{1}{\mathrm{i}\kappa} \sum_{m=1}^{N} \sum_{\gamma_{nm}} \left\langle S_{nm} \mathrm{div}_{\Gamma_m} \boldsymbol{J}_m, [\boldsymbol{v}]_{\gamma_{nm}} \right\rangle_{\gamma_{nm}} + \widetilde{a}_\gamma^{\pm}(\boldsymbol{J}, \boldsymbol{v}) \qquad \text{(symmetrization)}$$

$p_\gamma^{*}(\boldsymbol{J}, \boldsymbol{v})$ is function of $[\boldsymbol{J}]_{\gamma_{nm}}$ and $[\boldsymbol{v}]_{\gamma_{nm}}$ (penalization)

$$\textbf{(S)} \quad \widetilde{a}_\gamma^{-}(\boldsymbol{J}, \boldsymbol{v}) = -\frac{1}{\mathrm{i}\kappa} \sum_{n=1}^{N} \sum_{\gamma_{mn}} \left\langle S_{mn} \mathrm{div}_{\Gamma_n} \boldsymbol{v}_n, [\boldsymbol{J}]_{\gamma_{mn}} \right\rangle_{\gamma_{mn}}$$

$$\textbf{(AS)} \quad \widetilde{a}_\gamma^{+}(\boldsymbol{J}, \boldsymbol{v}) = +\frac{1}{\mathrm{i}\kappa} \sum_{n=1}^{N} \sum_{\gamma_{mn}} \left\langle S_{mn} \mathrm{div}_{\Gamma_n} \boldsymbol{v}_n, [\boldsymbol{J}]_{\gamma_{mn}} \right\rangle_{\gamma_{mn}}$$

- Requires more regularity : $\varepsilon = \frac{1}{2}$

$\boldsymbol{J}$ and $\boldsymbol{v} \in \bigoplus_{n=1}^{N} \mathrm{L}_{\mathrm{t}}^{2}(\mathrm{div}_{\Gamma_n}, \Gamma_n)$

$$\text{Find } \boldsymbol{J} \in \bigoplus_{m=1}^{N} \mathrm{L}^2_{\mathrm{t}}(\mathrm{div}_{\Gamma_m}, \Gamma_m) \text{ such that}$$

$$a_\Gamma(\boldsymbol{J},\boldsymbol{v}) + a^*_\gamma(\boldsymbol{J},\boldsymbol{v}) + p^*_\gamma(\boldsymbol{J},\boldsymbol{v}) = \ell(\boldsymbol{v}) \qquad \text{for any } \boldsymbol{v} \in \bigoplus_{n=1}^{N} \mathrm{L}^2_{\mathrm{t}}(\mathrm{div}_{\Gamma_n}, \Gamma_n)$$

- Empirical choice coming from Peng, Hiptmair and Shao (2016) : $L^2(\gamma)$-inner product

$$(\mathrm{L}^2) \qquad p^0_\gamma(\boldsymbol{J},\boldsymbol{v}) = \frac{\beta}{\kappa} \sum_{\gamma_{nm}} \langle [\boldsymbol{J}]_{\gamma_{nm}}, [\boldsymbol{v}]_{\gamma_{nm}} \rangle_{\gamma_{nm}}$$

where β have to be without dimension

  - requires more regularity (one more time !) to make sense : $\boldsymbol{J}$ and $\boldsymbol{v} \in \bigoplus_{n=1}^{N} \mathrm{H}^{\frac{1}{2}}_{\mathrm{t}}(\mathrm{div}_{\Gamma_n}, \Gamma_n)$

- A new penalization : $H^{-\frac{1}{2}}(\gamma)$-inner product (positive definite bilinear form)

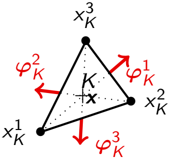
$$(\mathrm{H}^{-\frac{1}{2}}) \qquad p^{-\frac{1}{2}}_\gamma(\boldsymbol{J},\boldsymbol{v}) = \beta \sum_{\gamma_{nm}} \langle S_{\gamma_{nm}}[\boldsymbol{J}]_{\gamma_{nm}}, [\boldsymbol{v}]_{\gamma_{nm}} \rangle_{\gamma_{nm}}$$

where $\widetilde{S}_\gamma \lambda(\boldsymbol{x}) = \dfrac{1}{2\pi} \displaystyle\int_\gamma K_0(\kappa|\boldsymbol{x}-\boldsymbol{y}|)\lambda(\boldsymbol{y})\, \mathrm{d}\sigma_{\boldsymbol{y}}$ and β have to be without dimension

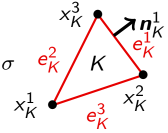
- Discretization space : Restrictions of Raviart-Thomas boundary elements of smallest degree

$$\boldsymbol{J} \approx \boldsymbol{J}^h \in \mathcal{V}^h \qquad \Longleftrightarrow \qquad \boldsymbol{J}^h = \sum_{m=1}^{N} \boldsymbol{J}^h_m \qquad \boldsymbol{J}^h_m \in \mathcal{V}^h_m \subset \mathrm{L}^2_t(\mathrm{div}_{\Gamma_m}, \Gamma_m)$$

where $\boldsymbol{J}^h_m(\boldsymbol{x}) = \sum_{K \in \Gamma^h_m} J^{h,m}_{K,1} \varphi^1_K(\boldsymbol{x}) + J^{h,m}_{K,2} \varphi^2_K(\boldsymbol{x}) + J^{h,m}_{K,3} \varphi^3_K(\boldsymbol{x})$ $\qquad \boldsymbol{x} \in \Gamma^h_m$ (triangulation of Γ_m)



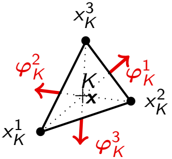
with $\varphi^i_K(\boldsymbol{x}) = \dfrac{1}{2|K|}\left(\boldsymbol{x} - x^i_K\right)$ and $J^{h,m}_{K,i} = \int_{e^i_K} \boldsymbol{J}^h_m \cdot \boldsymbol{n}^i_K \, \mathrm{d}\sigma$



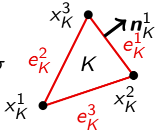
- Discretization space : Restrictions of Raviart-Thomas boundary elements of smallest degree

$$\boldsymbol{J} \approx \boldsymbol{J}^h \in \mathcal{V}^h \qquad \Longleftrightarrow \qquad \boldsymbol{J}^h = \sum_{m=1}^{N} \boldsymbol{J}_m^h \qquad \boldsymbol{J}_m^h \in \mathcal{V}_m^h \subset \mathrm{L}_t^2(\mathrm{div}_{\Gamma_m}, \Gamma_m)$$

where $\boldsymbol{J}_m^h(\boldsymbol{x}) = \sum_{K \in \Gamma_m^h} J_{K,1}^{h,m} \boldsymbol{\varphi}_K^1(\boldsymbol{x}) + J_{K,2}^{h,m} \boldsymbol{\varphi}_K^2(\boldsymbol{x}) + J_{K,3}^{h,m} \boldsymbol{\varphi}_K^3(\boldsymbol{x})$ $\qquad \boldsymbol{x} \in \Gamma_m^h$ (triangulation of Γ_m)



with $\boldsymbol{\varphi}_K^i(\boldsymbol{x}) = \dfrac{1}{2|K|}\left(\boldsymbol{x} - x_K^i\right)$ and $J_{K,i}^{h,m} = \displaystyle\int_{e_K^i} \boldsymbol{J}_m^h \cdot \boldsymbol{n}_K^i \, d\sigma$



Find $\boldsymbol{J}^h \in \mathcal{V}^h$ such that for any $\boldsymbol{v}^h \in \mathcal{V}^h$

(DG-EFIE) $\quad a_\Gamma^h(\boldsymbol{J}^h, \boldsymbol{v}^h) + a_\gamma^{*,h}(\boldsymbol{J}^h, \boldsymbol{v}^h) + p_\gamma^{*,h}(\boldsymbol{J}^h, \boldsymbol{v}^h) = \ell^h(\boldsymbol{v}^h)$

(DG-MFIE) $\quad b_\Gamma^h(\boldsymbol{J}^h, \boldsymbol{v}^h) = \widetilde{\ell}^h(\boldsymbol{v}^h)$ $\qquad$ (from continuous MFIE)

(DG-CFIE) $\quad \alpha$ **(DG-EFIE)** $+ (1-\alpha)$ **(DG-MFIE)** $\qquad$ (with $0 < \alpha < 1$)

- Use of GMRes solver (from CERFACS) with block-diagonal Jacobi preconditioning
  - Diagonal blocks correspond to matrices for individual subdomain
    - $\longrightarrow$ Flexibility in choosing subdomain solvers (LL^t, LU, Block low-rank, H-matrix, ...)
  - Off-diagonal blocks correspond to interactions between subdomains
    - $\longrightarrow$ Could be sped-up by compression techniques

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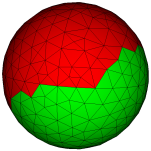
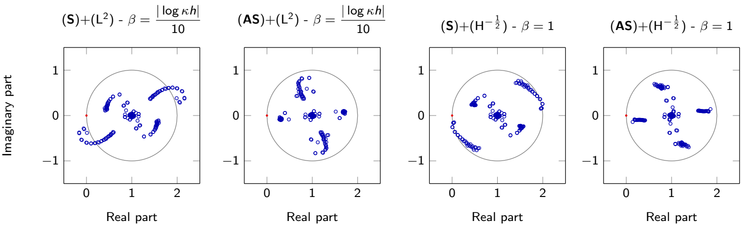
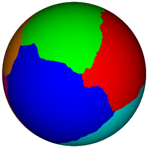
- Eingenvalue distribution : (**DG-CFIE**) simulation at 95 MHz with CFIE parameter $\alpha = 0.5$

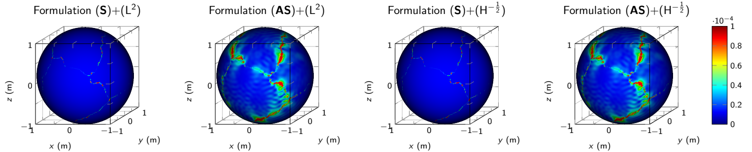




Outline

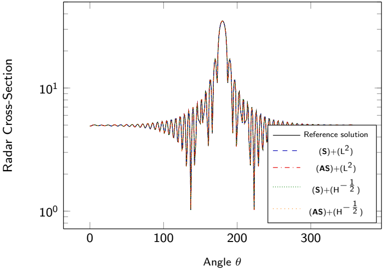
- Difference modulus between electric currents : (**DG-CFIE**) at $1.52\,\mathrm{GHz}$ - $h = \frac{\lambda}{15}$

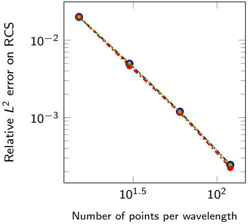


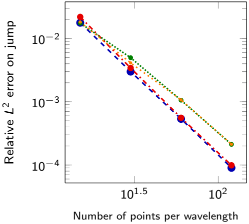


Maximal errors :	$8.681 \cdot 10^{-4}$	$4.920 \cdot 10^{-3}$	$1.269 \cdot 10^{-3}$	$7.004 \cdot 10^{-3}$

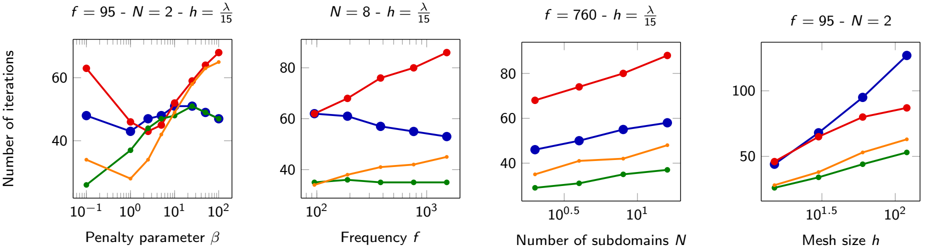
- Comparison of associated radar cross-sections and convergence ($95\,\mathrm{MHz}$ - 2 subdomains)



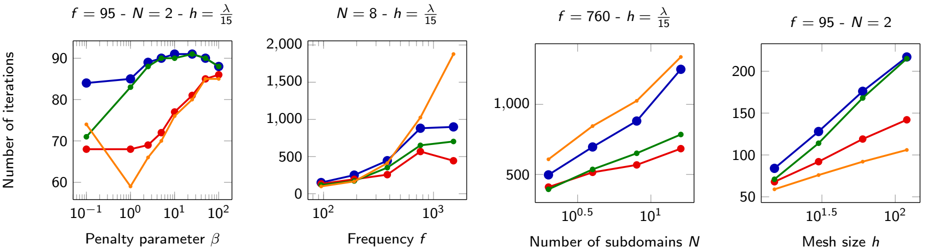


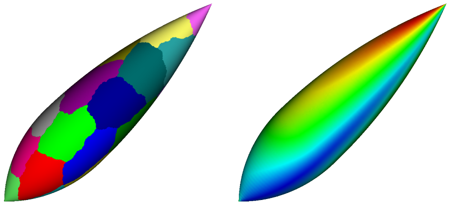


- **(DG-CFIE)** with $\alpha = 0.5$ (**S**)+(L^2) (**AS**)+(L^2) (**S**)+($H^{-\frac{1}{2}}$) (**AS**)+($H^{-\frac{1}{2}}$)



- **(DG-EFIE)**





- Dimensions : $2\lambda \times 2\lambda \times 6\lambda$
- Number of degrees of freedom : 273 312
- Number of subdomains : 24
- (**DG-CFIE**) simulation with $\alpha = 0.9$

- Comparison between CFIE+LU, CFIE+GMRes and DG-CFIE+GMRes

Formulation	Solver	Number of CPU	Factorization (in CPU·H)	Memory (in GB/MPI process)
CFIE	LU	3 328	5 018	11.43
DG-CFIE	GMRes	2 560	45	18.81

Formulation	Solver	Relative residual	Number of iterations	Convergence (in CPU·H)
CFIE	GMRes	10^{-3}	202	5 697
DG-CFIE	GMRes	10^{-3}	49	1 382
DG-CFIE	GMRes	10^{-6}	149	4 203

Conclusion

- Development and numerical analysis of a discontinuous Galerkin surface domain decomposition method for electromagnetic scattering by non-penetrable objects
  - Comparison of symmetric and anti-symmetric formulations
    - $\longrightarrow$ Symmetric formulation involves better accuracy on currents and preserves symmetry in EFIE
    - $\longrightarrow$ Anti-symmetric formulation involves better conditioning number in CFIE formulation
  - Comparison of L^2 and $H^{-\frac{1}{2}}$ interior penalty terms
    - $\longrightarrow$ L^2 penalization is robust with respect to the frequency but parameter β have to be calibrated
    - $\longrightarrow$ $H^{-\frac{1}{2}}$ penalization is more robust with respect to the discretization size
- Comparison with a boundary element method
  - Computational costs
    - $\longrightarrow$ Using a direct solver : large gain in factorization time
    - $\longrightarrow$ Using a preconditioned iterative solver : faster convergence for EFIE formulation
  - Accuracy : Jump does not pollute radar cross-sections

Perspectives

- Integration of *hp*-refinement
  - Non-conformal meshes
  - High-order boundary elements
- Integration of H-matrix formalism
- Extension to dielectric scatterers

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Thank you for your attention !