Fully adaptive lattice Boltzmann methods with error control based on multiresolution analysis

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Introduction and lattice Boltzmann schemes

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- **Stream**

$$f^\alpha(t+\Delta t,\boldsymbol{x}) = f^{\alpha,\star}(t,\boldsymbol{x}-\boldsymbol{c}_\alpha\Delta x).$$

How to devise the scheme and basic features of LBM

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- Fully explicit.
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- Strongly parallelizable.

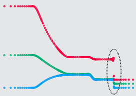
☹ **Disadvantages**

- Rely on a uniform Cartesian mesh and a particular time discretization.
- Only formal justification and a counter-intuitive way of imposing the physics.
- Stability conditions.
- Boundary conditions.

Why mesh adaptation . . .

Observation

In many problems, almost **all the variability** of the solution in concentrated in **few spots** (shocks or steep zones). This could be the solution of a lattice Boltzmann scheme.



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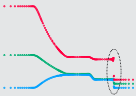

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- **Reduce the memory foot-print**.

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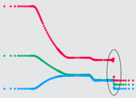

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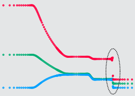

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Until now, two approaches are available:

- **Fixed meshed.** [Filippova and Hänel, 1998], [Lin and Lai, 2000], [Kandhai *et al.*, 2000], [Dupuis and Chopard, 2003].
- **Adaptive mesh refinement (AMR).** [Rohde *et al.*, 2006], [Fakhari and Lee, 2014], [Fakhari *et al.*, 2016].

Method	Simplicity	Problem independence	Optimization	Error control
Fixed mesh	■■	□□	□□	□□
AMR	■□	■□	■■	□□

Compared to the existing techniques, we want to achieve the following:

Constraints

- **Dynamically adapt** to the solution as time t advances (*vs.* fixed meshes).
- **Error control** by a small factor $0 < \epsilon \ll 1$ (*vs.* fixed meshes and AMR).
- **Problem independence** (*vs.* fixed meshes and AMR).
- **No scheme manipulation** (*vs.* fixed meshes and – some – AMR).

Multi-level grids and multiresolution

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$$\mathscr{L}_\ell := (C_{\ell,\boldsymbol{k}})_{\boldsymbol{k}}, \quad \text{with} \quad C_{\ell,\boldsymbol{k}} := \prod_{i=1}^{d} \left[2^{-\ell} k_i, 2^{-\ell}(k_i+1)\right],$$

for $\boldsymbol{k} = \{0,\dots,2^\ell - 1\}^d$ with space-step $\Delta x_\ell := 2^{-\ell}$, finest step $\Delta x = 2^{-\overline{L}}$ and $\Delta \ell = \overline{L} - \ell$: distance between the current level ℓ and the finest level $\overline{L}$. $\boldsymbol{x}_{\ell,\boldsymbol{k}} := 2^{-\ell}(\boldsymbol{k} + 1/2)$ is the cell center.

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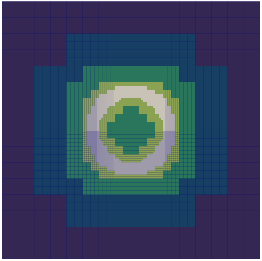

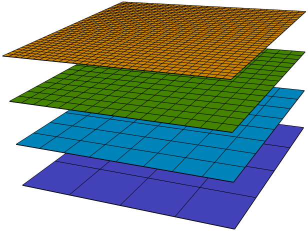

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From an abstract p.o.v., multiresolution, pioneered by [Daubechies, 1988] and [Mallat, 1989], is just a **decomposition** of data on a **wavelet basis** to study the local regularity of functions.

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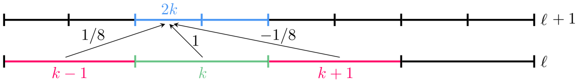

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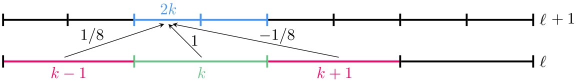

We use linear operators with **stencil size** γ of order $\mu = 2\gamma+1$ (extended for $d>1$ by tensor product [Bihari and Harten, 1997])

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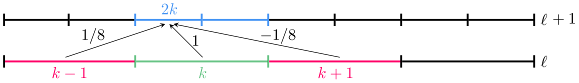

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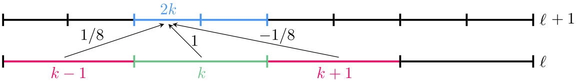

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We have the **isomorphism**

(Means finest level)		(Means coarsest level) + (Details each level)
$\overline{\boldsymbol{f}}^{\alpha}_{\overline{L}}$	$\Longleftrightarrow$	$\left(\overline{\boldsymbol{f}}^{\alpha}_{\underline{L}}, \overline{\boldsymbol{d}}^{\alpha}_{\underline{L}+1}, \dots, \overline{\boldsymbol{d}}^{\alpha}_{\overline{L}}\right)$

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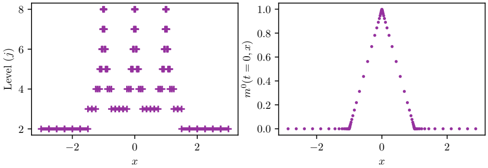
- **Less information** to store: detail to zero = erase the cell.
- **Reconstructing** information **controlling the error** (not possible with AMR).



Issue

The previous procedure is **static in time**! Our problem **evolves in time**.

[1] **Bellotti, Gouarin, Graille, Massot** - *Multidimensional fully adaptive lattice Boltzmann methods with error control based on multiresolution analysis* - Submitted to JCP - 2021 - `https://arxiv.org/abs/2103.02903`.

The fundamentals of multiresolution

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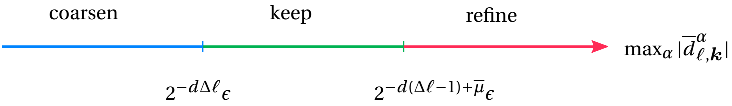

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$$\boxed{\begin{array}{llll} \text{Coarsen} & C_{\ell,\boldsymbol{k}} & \text{if} & \max_{\alpha} |\overline{d}^{\alpha}_{\ell,\boldsymbol{k}}| \leq 2^{-d\Delta\ell}\epsilon, \\ \text{Refine} & C_{\ell,\boldsymbol{k}} & \text{if} & \max_{\alpha} |\overline{d}^{\alpha}_{\ell,\boldsymbol{k}}| \geq 2^{-d(\Delta\ell-1)+\overline{\mu}}\epsilon \quad + \text{security cells.} \end{array}}$$

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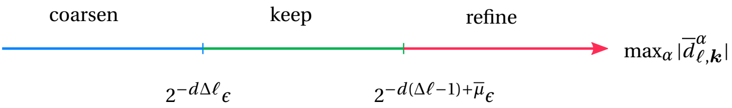

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For hyperbolic conservation laws, two basic principles guide the procedure are:

- **Propagation of information** at finite speed *via* advection phase: security cells.
- **Regularity loss** by non-linearity of the collision operator: refinement.

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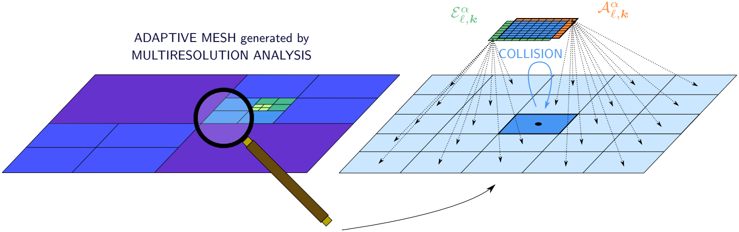
Adaptive lattice Boltzmann / multiresolution method

Adaptive lattice Boltzmann method



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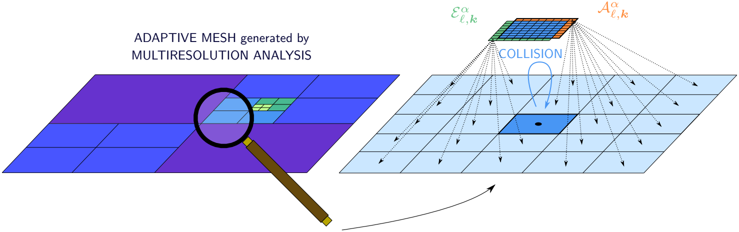

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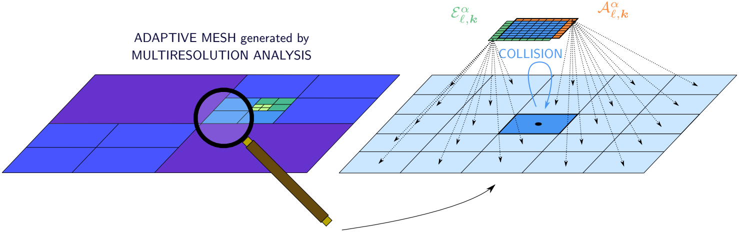

We have introduced[2]:

- **Collide** $\overline{\boldsymbol{f}}^{\star}_{\ell,\boldsymbol{k}}(t) = \boldsymbol{M}^{-1}\left((\boldsymbol{I}-\boldsymbol{S})\overline{\boldsymbol{m}}_{\ell,\boldsymbol{k}}(t) + \boldsymbol{S}\boldsymbol{m}^{\mathrm{eq}}(\overline{m}^{0}_{\ell,\boldsymbol{k}}(t),\dots)\right)$.

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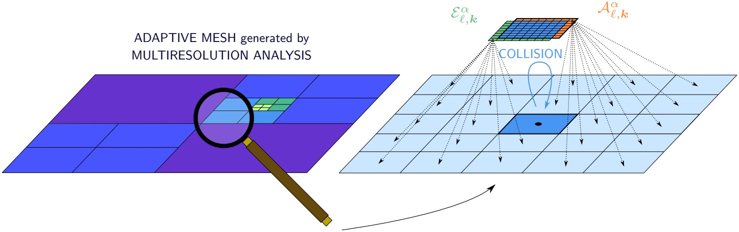

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- **Stream** $\overline{f}^{\alpha}_{\ell,\boldsymbol{k}}(t+\Delta t) = \overline{f}^{\alpha,\star}_{\ell,\boldsymbol{k}}(t) + \frac{1}{2^{d\Delta\ell}}\Big(\sum_{\overline{\boldsymbol{k}}\in\mathcal{E}^{\alpha}_{\ell,\boldsymbol{k}}} \widehat{\overline{f}}^{\alpha,\star}_{\overline{L},\overline{\boldsymbol{k}}}(t) - \sum_{\overline{\boldsymbol{k}}\in\mathcal{A}^{\alpha}_{\ell,\boldsymbol{k}}} \widehat{\overline{f}}^{\alpha,\star}_{\overline{L},\overline{\boldsymbol{k}}}(t)\Big)$,

  where we have taken

$$\mathcal{B}_{\ell,\boldsymbol{k}} = \{\boldsymbol{k}2^{\Delta\ell} + \boldsymbol{\delta} \, : \, \boldsymbol{\delta} \in \{0,\dots,2^{\Delta\ell}-1\}^{d}\},$$

$$\mathcal{E}^{\alpha}_{\ell,\boldsymbol{k}} = (\mathcal{B}_{\ell,\boldsymbol{k}} - \boldsymbol{c}_{\alpha}) \smallsetminus \mathcal{B}_{\ell,\boldsymbol{k}}, \qquad \mathcal{A}^{\alpha}_{\ell,\boldsymbol{k}} = \mathcal{B}_{\ell,\boldsymbol{k}} \smallsetminus (\mathcal{B}_{\ell,\boldsymbol{k}} - \boldsymbol{c}_{\alpha}).$$

In the figure, $\boldsymbol{c}_{\alpha} = (1,1)$. Why is it interesting?

---

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Achievements of the method

- Reduction in the **computational cost**, for at least three reasons:
  - Collision: Less evaluations of the non-linear equilibrium functions.
  - Collision: Less changes of variable to perform *via* $\mathbf{M}$.
  - Stream: Less numerical fluxes to compute. ($\sharp(\mathscr{E}^{\alpha}_{\ell,\boldsymbol{k}}) \propto 2^{(d-1)(\overline{L}-\ell)} \ll 2^{d(\overline{L}-\ell)}$).

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$$E[m^{\alpha}](t) = \frac{\sum_{\boldsymbol{k}\in\{0,\dots,2^{\overline{L}-1}\}^d} \Delta x \left| \widehat{\widehat{m}}^{\alpha}_{\overline{L},\boldsymbol{k}}(t) - m^{\mathrm{REF},\alpha}_{\overline{L},\boldsymbol{k}}(t) \right|}{\sum_{\boldsymbol{k}\in\{0,\dots,2^{\overline{L}-1}\}^d} \Delta x \left| m^{\mathrm{REF},\alpha}_{\overline{L},\boldsymbol{k}}(t) \right|},$$

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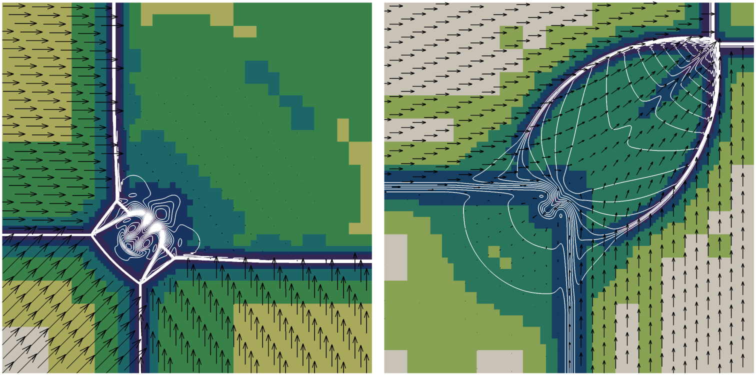
- (NEW↯) **No scheme modification**: everything done on the finest resolution *via* the reconstructions.
- (NEW↯) **Works for any scheme**.

(Quick) Assessment

2D non-isothermal Euler system

We consider the non-isothermal Euler system with the well-known Lax-Liu problem [Lax and Liu, 1998] simulated using a vectorial D2Q4 scheme[3]:

Colors: mesh levels – Contours: density field – Arrows: velocity field.



Dynamic adaptation, following shocks and fronts.

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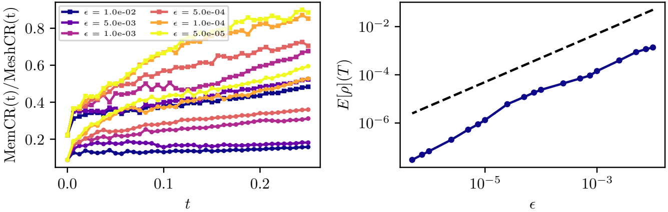
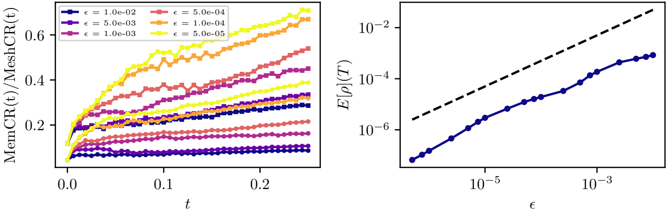
2D non-isothermal Euler system

$\underline{L} = 2$ and $\overline{L} = 8$



$\underline{L} = 2$ and $\overline{L} = 9$



- We effectively reach **high compression rates** (left) - low memory footprint.
- **Error control** by ϵ (right), showing the potetial of the multiresolution.

Promises kept but ... Need for a finer analysis

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Questions

Besides this nice control:

- How do we **perturb** the original system?
- What are the **physical phenomena** that we are still correctly modeling?
- Can multiresolution reduce these perturbations compared to traditional methods?

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Answer

Adapt the **available asymptotic analysis** (equivalent equations [Dubois, 2008]) used to analyze the lattice Boltzmann schemes.
To the best of our knowledge, first precise analysis of the effects of mesh adaptation on LBM schemes.

High accuracy and equivalent equations

Target expansion

We want to find the **maximum order of accuracy**[4] of our adaptive strategies according to the size of the prediction stencil γ. We adopt the point of view of Finite Differences [LEVEQUE, 2002]. When considered at the finest level $\overline{L}$

$$f^{\alpha}(t+\Delta t, x_{\overline{L},k}) = f^{\alpha,\star}(t, x_{\overline{L},k-c_{\alpha}}) = f^{\alpha,\star}(t, x_{\overline{L},k} - c_{\alpha}\Delta x).$$

[4] **Bellotti, Gouarin, Graille, Massot** - *High accuracy analysis of adaptive multiresolution-based lattice Boltzmann schemes via the equivalent equations.* - Submitted to the SMAI JCM - 2021 - `https://arxiv.org/abs/2105.13816`

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Thus we can apply a Taylor expansion to both sides of the equation, yielding

$$\sum_{s=0}^{+\infty} \frac{\Delta t^s}{s!} \partial_t^s f^{\alpha}(t, x_{\overline{L},k}) = \sum_{s=0}^{+\infty} \frac{(-c_\alpha \Delta x)^s}{s!} \partial_x^s f^{\alpha,\star}(t, x_{\overline{L},k})$$
$$= f^{\alpha,\star} - \underbrace{c_\alpha \Delta x \partial_x f^{\alpha,\star}}_{\text{Inertial term}} + \underbrace{\frac{c_\alpha^2 \Delta x^2}{2} \partial_{xx} f^{\alpha,\star}}_{\text{Diffusive term}} - \underbrace{\frac{c_\alpha^3 \Delta x^3}{6} \partial_x^3 f^{\alpha,\star}}_{\text{Dispersive term}} + \dots,$$

The right hand side is the **target expansion**. Indeed, the left hand side is always the same because the time-step Δt is fixed by the finest mesh.

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How to analyze our scheme? Assume, without loss of generality, that $\max_{\alpha} |c_{\alpha}| \leq 2$ and $\gamma \leq 1$.

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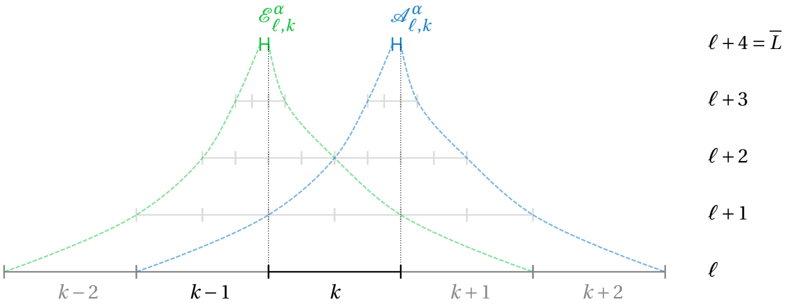
Recursion flattening



With a set of weights $(C^{\alpha}_{\Delta\ell,m})_{m=-2}^{m=+2} \subset \mathbb{R}$

$$\overline{f}^{\alpha}_{\ell,k}(t+\Delta t) = \overline{f}^{\alpha,\star}_{\ell,k}(t) + \frac{1}{2^{\Delta\ell}}\left(\sum_{\overline{k}\in\mathscr{E}^{\alpha}_{\ell,k}} \widehat{\widehat{\overline{f}}}^{\alpha,\star}_{\overline{L},\overline{k}}(t) - \sum_{\overline{k}\in\mathscr{A}^{\alpha}_{\ell,k}} \widehat{\widehat{\overline{f}}}^{\alpha,\star}_{\overline{L},\overline{k}}(t)\right)$$

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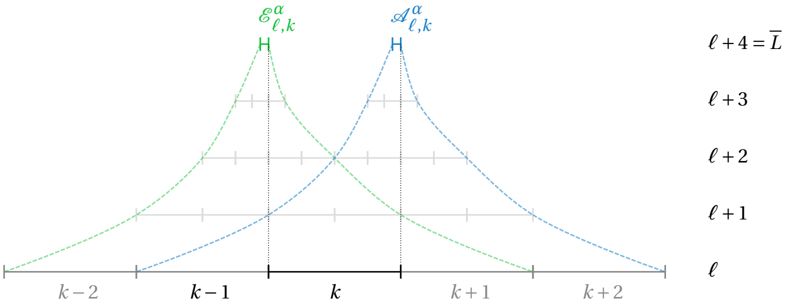

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$$=\overline{f}^{\alpha,\star}_{\ell,k}(t)+\frac{1}{2^{\Delta\ell}}\sum_{m=-2}^{+2}C^{\alpha}_{\Delta\ell,m}\overline{f}^{\alpha,\star}_{\ell,k+m}(t),$$

The advantage is that the pseudo-flux term can be developed using Taylor expansions adopting a Finite Difference point of view.

Expansion of the LBM-MR scheme

We can do the same expansion:

$$\sum_{s=0}^{+\infty}\frac{\Delta t^s}{s!}\partial_t^s f^{\alpha}(t,x_{\ell,k}) = f^{\alpha,\star}(t,x_{\ell,k}) + \sum_{s=0}^{+\infty}\left(\frac{(\Delta x_\ell)^s}{2^{\Delta\ell}s!}\left(\sum_{m=-2}^{+2} m^s C^{\alpha}_{\Delta\ell,m}\right)\partial_x^s f^{\alpha,\star}(t,x_{\ell,k})\right),$$

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The goal of this game is to match as much terms as possible of the target expansion: approximated physics and stability conditions as close as possible to that of the reference scheme at level $\overline{L}$ for the adaptive scheme at the local level of refinement ℓ.

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The goal of this game is to match as much terms as possible of the target expansion: approximated physics and stability conditions as close as possible to that of the reference scheme at level $\overline{L}$ for the adaptive scheme at the local level of refinement ℓ. These conditions are checked locally: we request them for any possible level.

$$\boxed{\sum_{m=-2}^{+2} C^{\alpha}_{\Delta \ell, m} = 0, \qquad \text{and} \qquad \sum_{m=-2}^{+2} m^s C^{\alpha}_{\Delta \ell, m} = \frac{(-c_{\alpha})^s}{2^{\Delta \ell (s-1)}}, \quad \text{for} \quad s \in \{1, 2, 3, \dots\} = \mathbb{N}^{\star},}$$

... of course for every α and for every $\Delta \ell$!!!

Apply the expansion to some scheme

In this presentation, we consider three schemes to adapt any lattice Boltzmann method:

- The **Haar scheme**: LBM-MR with $\gamma = 0$, thus

$$\widehat{\overline{f}}^{\alpha}_{\ell+1,2k+\delta} = \overline{f}^{\alpha}_{\ell,k}, \qquad (\textit{talis pater, qualis filius})_{\text{Abælardus}},$$

$$\text{thus} \qquad C^{\alpha}_{\Delta\ell,0} = -|c_{\alpha}|, \qquad C^{\alpha}_{\Delta\ell,-c_{\alpha}/|c_{\alpha}|} = |c_{\alpha}|.$$

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- The first **non-trivial wavelet scheme**: LBM-MR with $\gamma = 1$, thus

$$\widehat{f}^{\alpha}_{\ell+1,2k+\delta} = \overline{f}^{\alpha}_{\ell,k} + \frac{(-1)^{\delta}}{8}\left(\overline{f}^{\alpha}_{\ell,k+1} - \overline{f}^{\alpha}_{\ell,k-1}\right), \qquad (\textit{talis pater ac finitimi, qualis filius}),$$

$$\text{thus} \qquad \begin{pmatrix} C^{\alpha}_{\Delta\ell,-2} \\ C^{\alpha}_{\Delta\ell,-1} \\ C^{\alpha}_{\Delta\ell,0} \\ C^{\alpha}_{\Delta\ell,1} \\ C^{\alpha}_{\Delta\ell,2} \end{pmatrix} = \begin{pmatrix} 0 & -1/8 & 0 & 0 & 0 \\ 2 & 9/8 & 0 & -1/8 & 0 \\ 0 & 9/8 & 2 & 9/8 & 0 \\ 0 & -1/8 & 0 & 9/8 & 2 \\ 0 & 0 & 0 & -1/8 & 0 \end{pmatrix} \begin{pmatrix} C^{\alpha}_{\Delta\ell-1,-2} \\ C^{\alpha}_{\Delta\ell-1,-1} \\ C^{\alpha}_{\Delta\ell-1,0} \\ C^{\alpha}_{\Delta\ell-1,1} \\ C^{\alpha}_{\Delta\ell-1,2} \end{pmatrix}.$$

Apply the expansion to some scheme

In this presentation, we consider three schemes to adapt any lattice Boltzmann method:

- The **Haar scheme**: LBM-MR with $\gamma = 0$, thus

$$\widehat{f}^{\alpha}_{\ell+1,2k+\delta} = \overline{f}^{\alpha}_{\ell,k}, \qquad (\textit{talis pater, qualis filius})_{\text{Abælardus}},$$

$$\text{thus} \qquad C^{\alpha}_{\Delta\ell,0} = -|c_{\alpha}|, \qquad C^{\alpha}_{\Delta\ell,-c_{\alpha}/|c_{\alpha}|} = |c_{\alpha}|.$$

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- The **Lax-Wendroff scheme** by [Fakhari *et al.*, 2014]

$$C^{\alpha}_{\Delta\ell,0} = -\frac{|c_{\alpha}|^2}{2^{\Delta\ell}}, \quad C^{\alpha}_{\Delta\ell,-c_{\alpha}/|c_{\alpha}|} = \frac{|c_{\alpha}|}{2}\left(1 + \frac{|c_{\alpha}|}{2^{\Delta\ell}}\right), \quad C^{\alpha}_{\Delta\ell,c_{\alpha}/|c_{\alpha}|} = -\frac{|c_{\alpha}|}{2}\left(1 - \frac{|c_{\alpha}|}{2^{\Delta\ell}}\right).$$

  This is not a multiresolution scheme: we consider it for comparison purposes.

What is the expected accuracy?

We can prove that:

	Order	0	1 (Inertial)	2 (Diffusive)	3 (Dispersive)	4
	Condition	$\sum_m C^{\alpha}_{\Delta\ell,m} = 0$	$\sum_m mC^{\alpha}_{\Delta\ell,m} = -c_{\alpha}$	$\sum_m m^2 C^{\alpha}_{\Delta\ell,m} = \frac{c_{\alpha}^2}{2\Delta\ell}$	$\sum_m m^3 C^{\alpha}_{\Delta\ell,m} = -\frac{c_{\alpha}^3}{4\Delta\ell}$	$\sum_m m^4 C^{\alpha}_{\Delta\ell,m} = \frac{c_{\alpha}^4}{8\Delta\ell}$
Method	$\gamma = 0$	✓	✓	×	×	×
	$\gamma = 1$	✓	✓	✓	✓	×
	LW	✓	✓	✓	×	×

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	LW	✓	✓	✓	×	×

- **Original analysis** exploiting the structure of the multiresolution.
- **High fidelity to the desired physics**, beyond the existing approaches.
- Practically, **reliability** of the numerical method even in extreme situations.

Numerical simulations to assess the accuracy analysis

Points of emphasis

The previous analysis was valid for

- Smooth solutions.
- In the limit of small Δx_ℓ for every $\ell = \underline{L}, \dots, \overline{L}$.

The aim of the following numerical simulations is to **assess the previous approach** by showing that it provides a useful tool to *a priori* study the behavior of the adaptive scheme.

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- $\mathrm{E}_{\mathrm{ref}}$: error of the reference scheme (at $\overline{L}$) *vs.* exact solution. Intrinsic and sometimes converging for $\Delta x \to 0$.
- $\mathrm{E}_{\mathrm{adap}}^{\overline{L}}$: error of the adaptive scheme (at $\underline{L}$ but reconstructed) *vs.* exact solution at level $\overline{L}$.
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By triangle inequality $\mathrm{E}_{\mathrm{adap}}^{\overline{L}} \leq \mathrm{E}_{\mathrm{ref}} + \mathrm{D}_{\mathrm{adap}}$ and the plan is to make

$$\mathrm{D}_{\mathrm{adap}} \ll \mathrm{E}_{\mathrm{ref}}, \qquad \Rightarrow \qquad \mathrm{E}_{\mathrm{adap}}^{\overline{L}} \approx \mathrm{E}_{\mathrm{ref}},$$

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Is it reasonable? Yes, but no time to detail it.

1D Linear advection equation

- The aim of this test case is to validate our analysis in a case where:
  - **Convergent** reference scheme: $\mathrm{E}_{\mathrm{ref}} \to 0$ as $\Delta x \to 0$, see [DELLACHERIE, 2014], [CAETANO *et al.*, 2019].
  - **Only inertial** terms to model: we expect that all the schemes are suitable for this problem.
  - **Linear equilibria**: the collision strategy does not alter the quality of the method.

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$$\begin{cases} \partial_t u + \partial_x (V u) = 0, \\ u(t=0, x) = \frac{1}{(4\pi\nu t_0)^{1/2}} \exp\left(-\frac{|x|^2}{4\nu t_0}\right), \end{cases}$$

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$$\boldsymbol{M} = \begin{pmatrix} 1 & 1 \\ \lambda & -\lambda \end{pmatrix}, \qquad \boldsymbol{S} = \mathrm{diag}(0, s).$$

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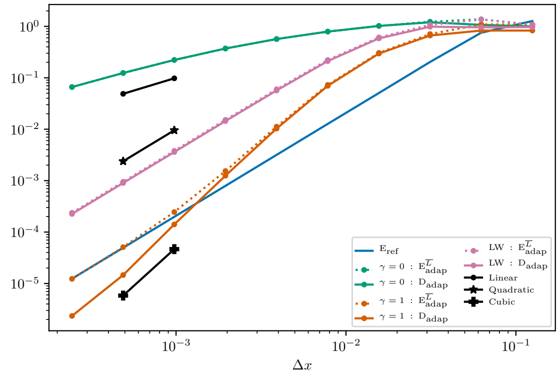
- We fix the level distance $\Delta\ell_{\min}$ and we increase $\overline{L}$ (thus reduce Δx, going towards convergence).

1D Linear advection equation: $\Delta\ell_{\min} = 2$ and $s = 2$

We have also treated $s = 1$, which means linear convergence $\mathrm{E}_{\mathrm{ref}} = \mathcal{O}(\Delta x)$.

Convergence of the different errors.



- $\gamma = 0$: $\mathrm{D}_{\mathrm{adap}} = \mathcal{O}(\Delta x) \gg \mathrm{E}_{\mathrm{ref}} = \mathcal{O}(\Delta x^2)$, thus $\mathrm{E}^{\overline{L}}_{\mathrm{adap}} \leq \mathrm{E}_{\mathrm{ref}} + \mathrm{D}_{\mathrm{adap}} = \mathcal{O}(\Delta x)$.
- $\gamma = 1$: $\mathrm{D}_{\mathrm{adap}} = \mathcal{O}(\Delta x^3) \ll \mathrm{E}_{\mathrm{ref}} = \mathcal{O}(\Delta x^2)$, thus $\mathrm{E}^{\overline{L}}_{\mathrm{adap}} \leq \mathrm{E}_{\mathrm{ref}} + \mathrm{D}_{\mathrm{adap}} = \mathcal{O}(\Delta x^2)$.
- Lax-Wendroff: $\mathrm{D}_{\mathrm{adap}} = \mathcal{O}(\Delta x^2) \sim \mathrm{E}_{\mathrm{ref}} = \mathcal{O}(\Delta x^2)$, thus $\mathrm{E}^{\overline{L}}_{\mathrm{adap}} \leq \mathrm{E}_{\mathrm{ref}} + \mathrm{D}_{\mathrm{adap}} = \mathcal{O}(\Delta x^2)$.

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$$m^{1,\mathrm{eq}} = V m^0, \qquad m^{2,\mathrm{eq}} = \kappa m^0$$

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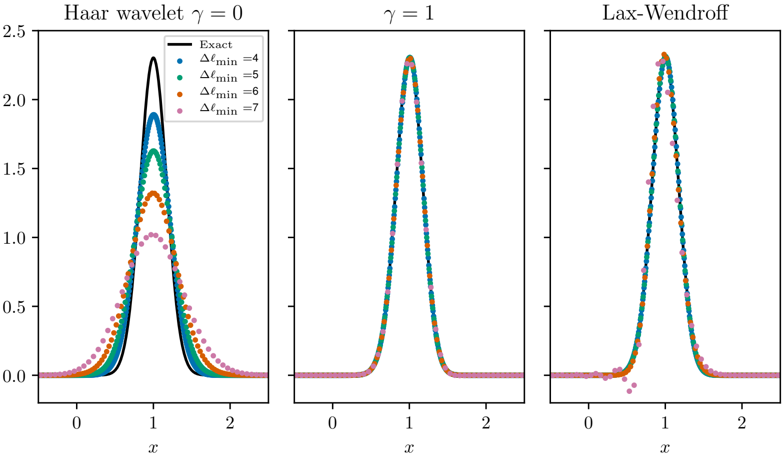
- We fix the maximal level $\overline{L}$ and we decrease the minimum level $\underline{L}$ (we increase $\Delta\ell_{\min}$).

1D Linear advection diffusion equation: $\overline{L} = 11$

Solution $u(T)$ at the final time for different $\Delta\ell_{\min}$.



- $\gamma = 0$: Wrong diffusion.
- $\gamma = 1$: Very good agreement.
- Lax-Wendroff: Spurious dispersive effects (third order).

Conclusions

What has been done (theoretically)

- Analysis based on the **equivalent equations** [Dubois, 2008] for the LBM-MR schemes.
- Find the **maximal order of compliance** of the adaptive scheme with the desired physics, depending on the prediction stencil γ.

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- **Good agreement** between the empirical behavior and the asymptotic analysis.

$$\partial_t u + \underbrace{\nabla\cdot(\boldsymbol{\varphi}(u))}_{\substack{\gamma=0\\ \gamma=1\\ \text{Lax-Wendroff}}} - \underbrace{\nabla\cdot(\boldsymbol{D}\nabla u)}_{\substack{\gamma=1\\ \text{Lax-Wendroff}}} = \underbrace{\text{H.O.Ts}}_{\gamma=1}.$$

  - The Lax-Wendroff scheme [Fakhari *et al.*, 2014]: minimal setting to use most of the LBM schemes. Unpredictable dispersive behaviors: threat to the stability.
  - The Haar scheme $\gamma = 0$ is almost unusable: it modifies the diffusive terms.
  - The LBM-MR scheme for $\gamma \geq 1$: most reliable of the analyzed schemes, both in terms of consistency and stability.
- If the solution is **singular**: adaptive mesh adaptation needed! (not really stressed here)

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- **Devised** a hybrid method to solve PDEs on time-evolving adapting meshes:

$$\text{Lattice Boltzmann methods} \quad \oplus \quad \text{Multiresolution}$$

[5] **Bellotti, Gouarin, Graille, Massot** - *Does the multiresolution lattice Boltzmann method allow to deal with waves passing through mesh jumps?* - Submitted to Comptes Rendus Mathématique - 2021 - `https://arxiv.org/abs/2105.12609`

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  It ensures

  - Time dynamic mesh adaptation.
  - Memory compression with shocks and fronts.
  - Reduced computational cost.
  - ↯ Totally problem and scheme independent.
  - ↯ Error control by a threshold $0 < \epsilon \ll 1$.
  - ↯ High (3rd) order accuracy w.r.t. the reference scheme: good approximation of the desired physics.
  - ↯ Realiability against mesh jumps[5]. Not presented here.

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- **Devised** an analysis based on the **equivalent equations** [Dubois, 2008] for the LBM-MR schemes. Confirming:
  - Very good agreement between the empirical behavior and the asymptotic analysis.
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Thank you for your attention!
Looking forward to receiving your questions!

Additional and backup material! Not a part of the presentation.

Points of emphasis

Remark

We are not interested in evaluating the quality of the multiresolution adaptation with respect to the parameter ϵ: we consider a uniform mesh at the lowest resolution $\underline{L}$.

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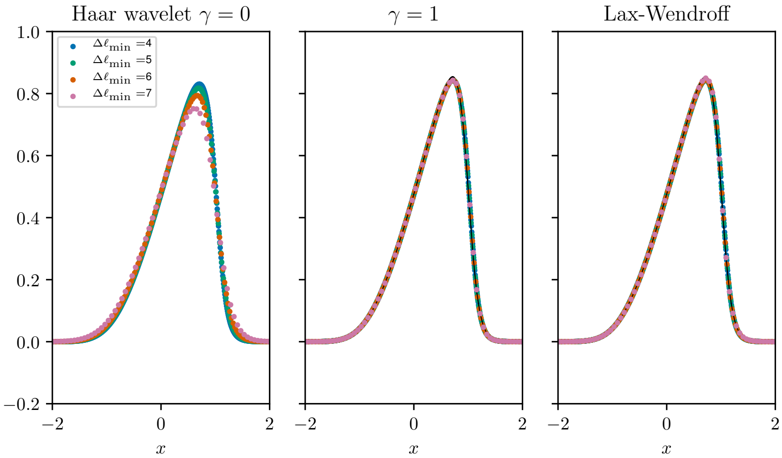
- **Worst case scenario** to undoubtedly prove the resilience of our numerical strategy. Similar scenarios can happen
  - when the mesh is updated using some **stiff** variable [Fakhari *et al.*, 2016] and [N'Guessan *et al.*, 2019] but we still want to achieve a good accuracy in the coarsely meshed areas for the non-stiff variables.
  - a fixed adapted mesh is used: [Filippova and Hänel, 1998] and many others.

Mesh adaptation: when is it needed?

Burgers equation: large diffusion.



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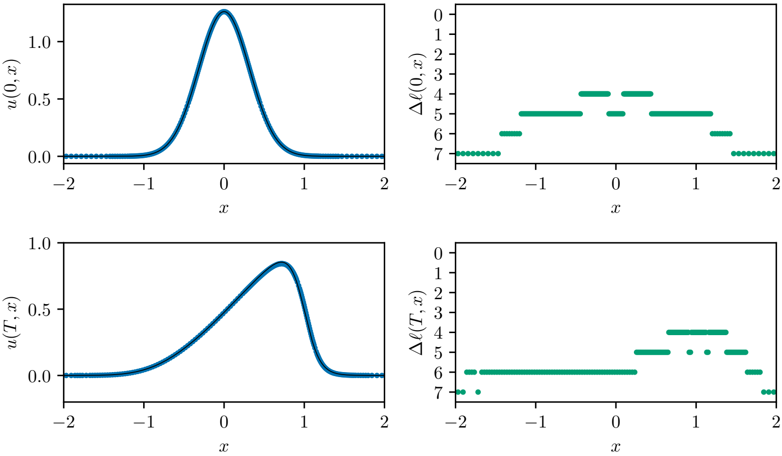

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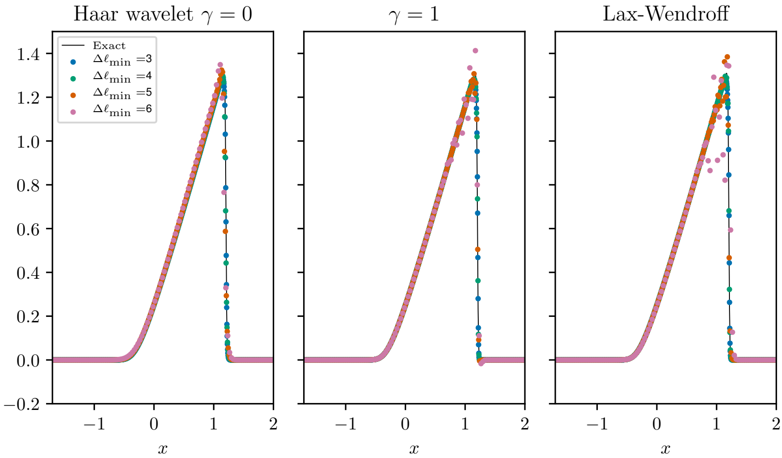

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