On the global solvability in time of the Navier-Stokes/Darcy coupled problem

for the fluid-porous inertial flows

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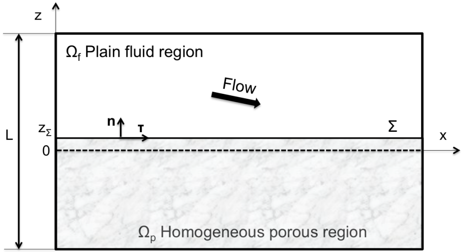
Coupling between free fluid and porous flows

Arbitrary flow direction in $\Omega := \Omega_f \cup \Sigma \cup \Omega_p$

$\Rightarrow$ Open problem for 2-D/3-D inertial flows (no global solvability)



Non-inertial flows with $0 < \phi_p < 1$: Stokes/Darcy-Brinkman problem

$$\begin{cases} \nabla\cdot v = 0 & \text{in } \Omega_f \cup \Omega_p, \\ -\mu\,\Delta v + \nabla p = \rho\, f & \text{in } \Omega_f, \\ -\dfrac{\mu}{\phi_p}\,\Delta v + \mu\, K_p^{-1} v + \nabla p = \rho\, f & \text{in } \Omega_p, \\ \text{suitable B.C., e.g.} \quad v = 0 & \text{on } \Gamma := \partial\Omega. \end{cases}$$

$\Rightarrow$ + physically relevant interface conditions on Σ at the macroscale... ?

Usual coupling for Navier-Stokes and Darcy flows

Ad-hoc extended Beavers-Joseph-Saffman's interface conditions

- From Beavers & Joseph (1967), Saffman (1971) and Jones (1973) : semi-empirical approach with experimental data for 1-D channel flow
- Jäger & Mikelić (2000, 2009), Brillard et al. (2013) : homogenization
- Mainly valid for the 1-D channel flow parallel to interface $\boldsymbol{\Sigma}$ see Angot et al. (2017, 2021), Eggenweiler & Rybak (2020, 2021)

$$\begin{cases} [\![\boldsymbol{v}\cdot\boldsymbol{n}]\!]_{\boldsymbol{\Sigma}} = \mathbf{0} \\ \boldsymbol{\tau_j}\cdot\left(\nabla\boldsymbol{v}+\nabla\boldsymbol{v}^T\right)^f_{\boldsymbol{\Sigma}}\cdot\boldsymbol{n} = \dfrac{\alpha_{\mathrm{bj}}}{\sqrt{K_p}}\,[\![\boldsymbol{v}\cdot\boldsymbol{\tau_j}]\!]_{\boldsymbol{\Sigma}}, \qquad \text{for } \; j=1,2 \qquad \text{on } \; \boldsymbol{\Sigma}=\boldsymbol{\Sigma_t}. \\ \boldsymbol{n}\cdot[\![\boldsymbol{\sigma}(\boldsymbol{v},\boldsymbol{p})\cdot\boldsymbol{n}]\!]_{\boldsymbol{\Sigma}} = \mathbf{0} \end{cases}$$

- Saffman's approx. : $[\![\boldsymbol{v}\cdot\boldsymbol{\tau}]\!]_{\boldsymbol{\Sigma}} := (\boldsymbol{v}^f - \boldsymbol{v}^p)_{\boldsymbol{\Sigma}}\cdot\boldsymbol{\tau} \approx \boldsymbol{v}^f\cdot\boldsymbol{\tau}$, not always valid
- $\Rightarrow$ In contradiction with non-zero stress vector jump derived by asymptotic modeling theory of PhA., Goyeau and Ochoa-Tapia (2017)
- $\Rightarrow$ Only linear conditions in contradiction with the nonlinear ones derived by PhA., Goyeau and Ochoa-Tapia (2021)

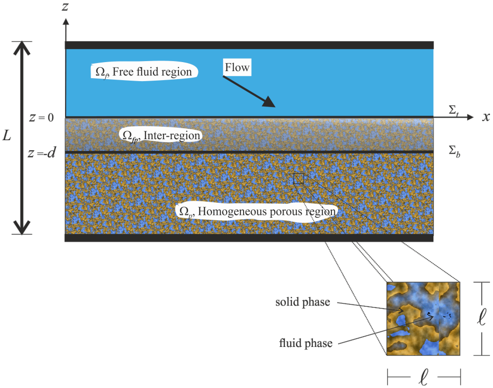
Outline

1. *Asymptotic models at a fluid-porous interface*
2. *Nonlinear asymptotic models for the inertial flows*
3. *Analysis of the Navier-Stokes/Darcy problem*
4. *Conclusion and perspectives*

Asymptotic modeling and related approximations

New theory with given interface position z_Σ ($\neq$ upscaling theories)

See [PhA., Goyeau and Ochoa-Tapia, Phys. Rev. E, 2017]



Single-domain continuum modeling with heterogeneous inter-region $\boldsymbol{\Omega_{fp}}$

- Dimensional reduction of $\boldsymbol{\Omega_{fp}}$ to a dividing surface $\boldsymbol{\Sigma}$ (as for fractures)
- Average over $\boldsymbol{d}$ generalized Brinkman (variable porosity & permeability)
- Asymptotic analysis at $\boldsymbol{O(d/L)}$ with $\boldsymbol{\delta := d/L \ll 1}$ since $\boldsymbol{d = O(\sqrt{K_p})}$
- $\Rightarrow$ Sets of jump interface conditions depending on $\boldsymbol{0 \leq \xi := |z_\Sigma|/d \leq 1}$

Approximations at the first-order in $O(d/L)$

A key result extensively used

Lemma (Approximation of the generalized average over thickness d)

Let $\psi : [-d/2, d/2] \to \mathbb{R}$ *be a continuously differentiable function and let* $w : [-d/2, d/2] \to \mathbb{R}$ *be Lebesgue-integrable.*
Then we have :

$$\int_{-d/2}^{d/2} w(x)\,\psi(x)\,\mathrm{d}x$$

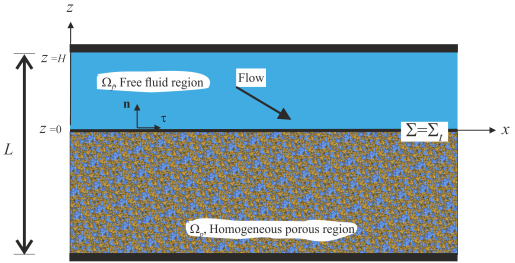
$$= d\,\langle w\rangle\,\frac{\psi(-d/2)+\psi(d/2)}{2} + O\left(\|\psi'\|_\infty \langle |w|\rangle\, d^2\right)$$

$$= d\,\langle w\rangle\,\overline{\psi}_\Sigma + O\left(\|\psi'\|_\infty \langle |w|\rangle\, d^2\right).$$

$\Rightarrow$ Averaging over d the product of functions as Darcy drag : $\mu\, K^{-1} v$
$\Rightarrow$ Averaging over d the nonlinear terms coming from Navier-Stokes or Dupuit-Forchheimer

Asymptotic interface models up to $O(d/L)$

Jump interface conditions on $\Sigma = \Sigma_t$ ($\xi = 0$) for non-inertial flow

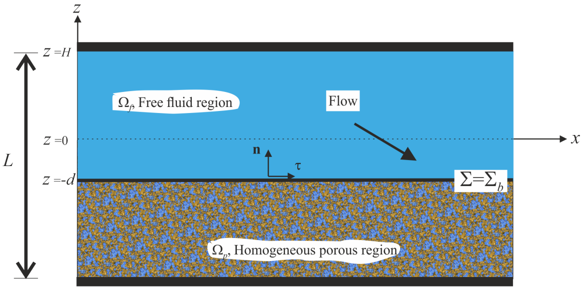


$$\left\{\begin{aligned} [\![\boldsymbol{v}\cdot\boldsymbol{n}]\!]_{\Sigma} &= 0, \\ \boldsymbol{\sigma}_v^f(\boldsymbol{v})\cdot\boldsymbol{n}_{\Sigma} &= \frac{\mu}{\sqrt{K_p}}\,\alpha_{\Sigma}\,[\![\boldsymbol{v}]\!]_{\Sigma}, \\ [\![\boldsymbol{\sigma}(\boldsymbol{v},p)\cdot\boldsymbol{n}]\!]_{\Sigma} &= \frac{\mu}{\sqrt{K_p}}\,\boldsymbol{\beta}_{\Sigma}\cdot\boldsymbol{v}_{\Sigma}^f - \boldsymbol{f}_{\Sigma} \end{aligned}\right. \qquad \text{on } \Sigma = \Sigma_t.$$

- $\alpha_\Sigma > 0$: coefficient of tangential velocity slip (only scalar)
- $\beta_\Sigma \geq 0$: tensor (symmetric semi-positive definite) of Darcy friction
- $\Rightarrow$ Multi-directional 2-D/3-D flows with anisotropic effects of microstructure for both Stokes/Darcy-Brinkman or Stokes/Darcy models

Asymptotic interface models up to $O(d/L)$

Jump interface conditions on $\Sigma = \Sigma_b$ ($\xi = 1$) for non-inertial flow



$$\begin{cases} [\![v]\!]_\Sigma = 0, \qquad i.e. \quad v^f_\Sigma = v^p_\Sigma := v_\Sigma \\ [\![\sigma(v,p)\cdot n]\!]_\Sigma = \dfrac{\mu}{\sqrt{K_p}}\beta_\Sigma \cdot v_\Sigma - f_\Sigma \end{cases} \quad \text{on } \Sigma = \Sigma_b.$$

- Velocity continuity on Σ_b for Stokes/Darcy-Brinkman and Stokes/Darcy
- $\beta_\Sigma > 0$: tensor (symmetric positive definite) of Darcy friction
- $\Rightarrow$ Multi-directional 2-D/3-D flows with anisotropic effects of microstructure for both Stokes/Darcy-Brinkman or Stokes/Darcy models

The incompressible inertial viscous flow

Navier-Stokes/Darcy-Forchheimer model $(0 < \phi_p \leq 0.95)$

$\Rightarrow$ Volume-averaged Navier-Stokes equations in $\boldsymbol{\Omega_{fp}}$ with $\boldsymbol{\phi_p \leq \phi \leq 1}$
All the terms must be kept in $\boldsymbol{\Omega_{fp}}$ $\Rightarrow$ coherency when $\boldsymbol{\phi \to 1}$ or $\boldsymbol{\phi \to \phi_p}$
See [PhA., Goyeau and Ochoa-Tapia, Adv. Water Res., 2021]

$$\begin{aligned}
\boldsymbol{\nabla \cdot v} &= \mathbf{0} && \text{in } \boldsymbol{\Omega}, \\
\boldsymbol{\nabla \cdot (\rho\, v \otimes v) - \mu\, \Delta v + \nabla p} &= \boldsymbol{\rho\, f} && \text{in } \boldsymbol{\Omega_f}, \\
\boldsymbol{\frac{1}{\phi} \nabla \cdot \left(\frac{\rho}{\phi}\, v \otimes v \right) - \nabla \cdot \sigma(v, p)} & \\
\boldsymbol{+ \mu\, K^{-1} \cdot v + \frac{\rho}{\sqrt{K_p}}\, |v|\, \kappa(\phi) \cdot v} &= \boldsymbol{\rho\, f} && \text{in } \boldsymbol{\Omega_{fp}}, \\
\boldsymbol{\mu\, K_p^{-1} \cdot v + \frac{\rho}{\sqrt{K_p}}\, |v|\, \kappa(\phi_p) \cdot v + \nabla p} &= \boldsymbol{\rho\, f} && \text{in } \boldsymbol{\Omega_p},
\end{aligned}$$

where :

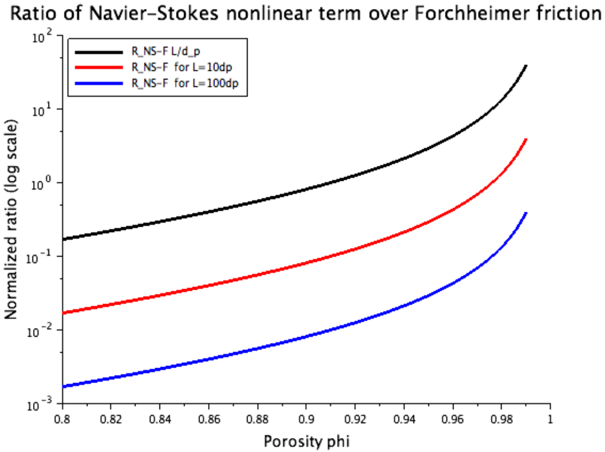
- $\boldsymbol{\kappa(\phi)}$: Dupuit-Forchheimer's inertial tensor (strong inertia regime)
- stress tensor for a Newtonian fluid in $\boldsymbol{\Omega_f}$ or $\boldsymbol{\Omega_{fp}}$:

$$\boldsymbol{\sigma(v, p) := \frac{\mu}{\phi} (\nabla v + \nabla v^T) - p\, I},$$

$$\text{with} \quad \boldsymbol{\phi = 1} \ \text{ in } \ \boldsymbol{\Omega_f}, \quad \boldsymbol{\phi = \phi_p} \ \text{ in } \ \boldsymbol{\Omega_p}.$$

- stress tensor for Darcy's law in $\boldsymbol{\Omega_p}$: $\boldsymbol{\sigma^p(v, p) := -p\, I}$

The nonlinear interface model for the inertial flow



Ratio of Navier-Stokes nonlinear term over Forchheimer's inertial drag

$$R_{NS/F} = O(\sqrt{K(\phi)}/\phi^2 L\, \kappa(\phi))$$

$\Rightarrow$ Navier's nonlinear term can be neglected in $\boldsymbol{\Omega_p}$ for $\boldsymbol{\phi_p \leq 0.95}$, but not in $\boldsymbol{\Omega_{fp}}$ ($\boldsymbol{\phi_p \leq \phi \leq 1}$) since the two nonlinear terms are then of the same order.

The nonlinear interface model for the inertial flow

Navier-Stokes/Darcy-Forchheimer macroscopic model

+ the nonlinear interface model at $\Sigma = \Sigma_m$ (middle of Ω_{fp}) up to $O(d/L)$ with a surface force $f_\Sigma := d\,\langle \rho\, f \rangle$ on Σ :

$$\begin{cases} [\![v\cdot n]\!]_\Sigma = 0, \\ \overline{\sigma_v(v)\cdot n}_\Sigma = \dfrac{\mu}{\sqrt{K_p}}\,\alpha_\Sigma\,[\![v]\!]_\Sigma, \\ [\![\sigma(v,p)\cdot n]\!]_\Sigma = \dfrac{\mu}{\sqrt{K_p}}\,\beta_\Sigma\cdot\overline{v}_\Sigma \\ \qquad + \dfrac{1}{2}\,\rho\, v\cdot n\, v^f + \rho\,\overline{|v|}_\Sigma\,\lambda_\Sigma\cdot\overline{v}_\Sigma - f_\Sigma \end{cases} \quad \text{on } \Sigma = \Sigma_m.$$

where $\overline{\psi}_\Sigma := \left(\psi^f + \psi^p\right)_\Sigma / 2$ (arithmetic mean of traces on Σ)

- $\Rightarrow$ Non-inertial interface model on Σ_m
- + two additional nonlinear terms coming from inertia in Ω_{fp} ($\lambda_\Sigma > 0$)
- $\dfrac{1}{2}\,\rho\, v\cdot n\, v^f \neq \dfrac{1}{2}\,\rho\,|v^f|^2\, n$ (suggested conjecture of Rivière et al., 2008)
- $\Rightarrow$ To our knowledge, this nonlinear multi-dimensional model is the first proposed in the literature for the inertial flow with global dissipation

The nonlinear interface model for the inertial flow

Navier-Stokes/Darcy-Forchheimer macroscopic model

+ the nonlinear interface model at $\boldsymbol{\Sigma = \Sigma_b}$ ($\boldsymbol{\xi = 1}$) up to $\boldsymbol{O(d/L)}$
with a surface force $\boldsymbol{f_\Sigma := d \, \langle \rho \, f \rangle}$ on $\boldsymbol{\Sigma}$:

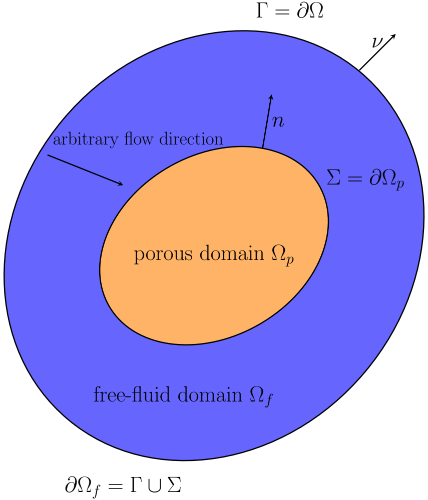
$$\left\{ \begin{aligned} [\![v]\!]_\Sigma &= 0, \qquad i.e. \quad v_\Sigma^f = v_\Sigma^p := v_\Sigma \\ [\![\sigma(v,p)\cdot n]\!]_\Sigma &= \frac{\mu}{\sqrt{K_p}} \, \beta_\Sigma \cdot v_\Sigma \\ &\quad + \frac{1}{2} \, \rho \, v\cdot n \, v_\Sigma + \rho \, |v_\Sigma| \, \lambda_\Sigma \cdot v_\Sigma - f_\Sigma \end{aligned} \right. \qquad \text{on } \Sigma = \Sigma_b.$$

- $\Rightarrow$ Non-inertial interface model on $\boldsymbol{\Sigma_b}$
  stress jump interface conditions with velocity continuity
- + two additional nonlinear terms coming from inertia in $\boldsymbol{\Omega_{fp}}$ ($\boldsymbol{\lambda_\Sigma > 0}$)
- $\Rightarrow$ Only the parameter $\boldsymbol{\beta_\Sigma}$ of the non-inertial interface model is required at $\boldsymbol{\Sigma = \Sigma_b}$ ($\boldsymbol{\xi = 1}$) since $\boldsymbol{\lambda_\Sigma \simeq \beta_\Sigma \, \kappa(\phi_p)}$.

Unsteady Navier-Stokes/Darcy coupled problem

Flow configuration in fluid-porous domain $\boldsymbol{\Omega := \Omega_f \cup \Sigma \cup \Omega_p \subset \mathbb{R}^d}$ *(d ≤ 3)*

$\boldsymbol{\Omega}$: open bounded connected domain with Lipschitz boundary $\boldsymbol{\Gamma := \partial\Omega}$
$\boldsymbol{\Sigma := \partial\Omega_p}$: closed smooth surface (of class $\mathscr{C}^{\mathbf{1,1}}$) or $\boldsymbol{\Omega_p}$ a convex domain (to avoid unnecessary technicalities...)



Unsteady Navier-Stokes/Darcy coupled problem

Coupling with nonlinear stress jump interface conditions on $\Sigma = \Sigma_b$

$$
\begin{cases}
\nabla \cdot v = 0 & \text{in } (0,T) \times \Omega, \\
\partial_t v^f + (\nabla \times v^f) \wedge v^f - \mu \, \Delta v^f + \nabla \left(p^f + \frac{1}{2} |v^f|^2 \right) = f^f & \text{in } (0,T) \times \Omega_f, \\
\varepsilon \, \partial_t v^p + \mu \, K^{-1} \, v^p + \nabla p^p = f^p & \text{in } (0,T) \times \Omega_p, \\
v^f = 0 & \text{on } (0,T) \times \Gamma, \\
v(t=0) = v_0 & \text{in } \Omega,
\end{cases}
$$

with the data :

- constant mass density $\rho = 1$ and dynamic viscosity $\mu > 0$
- constant intrinsic permeability tensor K (symmetric and positive definite), possibly time-dependent : $K \in L^2(]0,T[; \mathbb{R}^{d \times d})$ s.t. with some $k_m > 0$: $(K^{-1}(t) \, y) \cdot y \geq k_m \, |y|^2$ for all $y \in \mathbb{R}^d$ and almost every $t \in (0,T)$.
- body forces : $f^f \in L^2(0,T; L^2(\Omega_f)^d)$ and $f^p \in L^2(0,T; H^1(\Omega)^d)$
- small parameter $\varepsilon > 0$ possibly going to zero

Nonlinear stress jump interface conditions on $\Sigma = \Sigma_b$

Cauchy stress tensor in $\boldsymbol{\Omega_f}$ or $\boldsymbol{\Omega_p}$:

$$\begin{cases} \boldsymbol{\sigma}^f(\boldsymbol{v},p) := \boldsymbol{\sigma}_v^f(\boldsymbol{v}) - p\,\boldsymbol{I}, & \text{where} \quad \boldsymbol{\sigma}_v^f(\boldsymbol{v}) := 2\,\mu\,\boldsymbol{D}(\boldsymbol{v}) \qquad \text{in } \boldsymbol{\Omega_f}, \\ & \text{with} \quad \boldsymbol{D}(\boldsymbol{v}) := \frac{1}{2}\left(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T\right), \\ \boldsymbol{\sigma}^p(\boldsymbol{v},p) := -p^p\,\boldsymbol{I}, & \qquad \text{in } \boldsymbol{\Omega_p}, \end{cases}$$

where $\boldsymbol{\sigma}_v^f(\boldsymbol{v})$ is the viscous stress tensor and $\boldsymbol{D}(\boldsymbol{v})$ is the strain rate tensor (symmetric part of $\nabla \boldsymbol{v}$), $\boldsymbol{I}$ being the unit tensor.

$$\begin{cases} [\![\boldsymbol{v}]\!]_{\Sigma} = 0, \qquad \text{i.e.} \quad \boldsymbol{v}_{\Sigma}^f = \boldsymbol{v}_{\Sigma}^p := \boldsymbol{v}_{\Sigma} \\ [\![\boldsymbol{\sigma}(\boldsymbol{v},p)\cdot\boldsymbol{n}]\!]_{\Sigma} = \dfrac{\mu}{\sqrt{K}}\,\boldsymbol{\beta}_{\Sigma}\,\boldsymbol{v}_{\Sigma} + \dfrac{1}{2}\,\boldsymbol{v}\cdot\boldsymbol{n}\,\boldsymbol{v}_{\Sigma} + |\boldsymbol{v}_{\Sigma}|\,\boldsymbol{\lambda}_{\Sigma}\,\boldsymbol{v}_{\Sigma} - \boldsymbol{f}_{\Sigma} \end{cases} \quad \text{on } \Sigma = \Sigma_b,$$

where :

- $K := \|\boldsymbol{K}\|_2$ (or another permeability reference)
- given external surfacic force on Σ : $\boldsymbol{f}_{\Sigma} \in L^2(0,T;H^{-1/2}(\Sigma)^d)$
- stress jump friction tensor (symmetric and uniformly positive definite) : $\boldsymbol{\beta}_{\Sigma} \in L^2(0,T;L^{\infty}(\Sigma)^{d\times d})$
- inertial friction tensor (uniformly bounded and positive definite, possibly symmetric) : $\boldsymbol{\lambda}_{\Sigma} \in L^{\infty}((0,T)\times\Sigma)^{d\times d}$

Leray's type weak problem with divergence-free test functions

Functional setting :

$$\begin{aligned} \boldsymbol{H} &:= \left\{ \boldsymbol{u} \in L^2(\Omega)^d ;\ \nabla\cdot\boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u}\cdot\boldsymbol{\nu} = 0 \text{ on } \Gamma \right\}, \\ \boldsymbol{V} &:= \left\{ \boldsymbol{u} \in H_0^1(\Omega)^d ;\ \nabla\cdot\boldsymbol{u} = 0 \text{ in } \Omega \right\}, \\ \boldsymbol{V_f} &:= \left\{ \boldsymbol{u} \in H_{0,\Gamma}^1(\Omega_f)^d ;\ \nabla\cdot\boldsymbol{u} = 0 \text{ in } \Omega_f \right\}. \end{aligned}$$

Equivalent weak transmission problem : looking for a velocity solution $\boldsymbol{v} \in L^2(0,T;\boldsymbol{V})$ such that $\boldsymbol{v}'^f := \partial_t \boldsymbol{v}^f \in L^1(0,T;H^{-1}(\Omega_f)^d)$

$$\left\{ \begin{aligned} &\left\langle \partial_t \boldsymbol{v}^f, \boldsymbol{\varphi}^f \right\rangle_{V_f',V_f} + \int_{\Omega_f} ((\nabla \times \boldsymbol{v})\wedge \boldsymbol{v})\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \int_{\Omega_f} \mu\,\nabla\boldsymbol{v} : \nabla\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \int_{\Omega_p} \mu\,(K^{-1}\boldsymbol{v})\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} \\ &\quad + \int_\Sigma \frac{\mu}{\sqrt{K}}(\beta_\Sigma\,\boldsymbol{v})\cdot\boldsymbol{\varphi}\,\mathrm{d}s + \int_\Sigma |\boldsymbol{v}_\Sigma|\,(\lambda_\Sigma\,\boldsymbol{v})\cdot\boldsymbol{\varphi}\,\mathrm{d}s \\ &\quad + \frac{1}{2}\int_\Sigma \left(\boldsymbol{v}\cdot\boldsymbol{n}\,\boldsymbol{v}\cdot\boldsymbol{\varphi} - |\boldsymbol{v}|^2\,\boldsymbol{\varphi}\cdot\boldsymbol{n} \right)\mathrm{d}s \qquad \Rightarrow \text{ vanishing term when } \boldsymbol{\varphi} = \boldsymbol{v} \\ &\quad = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{f}_\Sigma, \boldsymbol{\varphi} \rangle_{-1/2,\Sigma}, \qquad \text{for all } \boldsymbol{\varphi} \in \boldsymbol{V}, \end{aligned} \right.$$

supplemented with the initial condition $\boldsymbol{v}(t=0) = \boldsymbol{v}_0$.

It makes sense for any $\boldsymbol{v}^f(t) \in H^1(\Omega_f)^d$ and $\boldsymbol{v}^p(t) \in H^1(\Omega_p)^d$ (H^2-elliptic regularity of pressure in Ω_p) with Sobolev continuous imbeddings for $d \leq 3$: $H^1(\Omega_f)^d \hookrightarrow L^6(\Omega_f)^d$ and $H^{1/2}(\Sigma)^d \hookrightarrow L^4(\Sigma)^d$

Unsteady Navier-Stokes/Darcy coupled problem

Linearly implicit discrete time scheme with time-step $\delta t > 0$

u^k : the time approximate of any function $u(t)$ at time $t_k := k\,\delta t$

Starting from $v^0 = v_0$, the Euler implicit scheme reads for all $k \in \mathbb{N}$ such that $(k+1)\,\delta t \leq T$:

$$\left\{\begin{aligned}
&\int_{\Omega_f} \frac{v_\varepsilon^{k+1} - v_\varepsilon^k}{\delta t}\cdot\varphi\,\mathrm{d}x + \varepsilon \int_{\Omega_p} \frac{v_\varepsilon^{k+1} - v_\varepsilon^k}{\delta t}\cdot\varphi\,\mathrm{d}x + \int_{\Omega_f} \Big((\nabla \times v_\varepsilon^k)\wedge v_\varepsilon^{k+1}\Big)\cdot\varphi\,\mathrm{d}x \\
&\qquad + \int_{\Omega_f} \mu\,\nabla v_\varepsilon^{k+1} : \nabla\varphi\,\mathrm{d}x + \int_{\Omega_p} \mu\,(K^{-1,(k+1)} v_\varepsilon^{k+1})\cdot\varphi\,\mathrm{d}x \\
&\qquad + \int_\Sigma \frac{\mu}{\sqrt{K}}\,(\beta_\Sigma^{k+1}\,v_\varepsilon^{k+1})\cdot\varphi\,\mathrm{d}s + \int_\Sigma |v_\varepsilon^k|\,(\lambda_\Sigma^{k+1}\,v_\varepsilon^{k+1})\cdot\varphi\,\mathrm{d}s \\
&\qquad + \frac{1}{2}\int_\Sigma \Big(v_\varepsilon^{k+1}\cdot n\,v_\varepsilon^k\cdot\varphi - (v_\varepsilon^k\cdot v_\varepsilon^{k+1})\,\varphi\cdot n\Big)\,\mathrm{d}s \quad (= 0 \ \text{ with } \ \varphi = v_\varepsilon^{k+1}) \\
&\qquad = \int_\Omega f^{k+1}\cdot\varphi\,\mathrm{d}x + \Big\langle f_\Sigma^{k+1}, \varphi\Big\rangle_{-1/2,\Sigma}, \qquad \text{for all } \ \varphi \in V.
\end{aligned}\right.$$

Unsteady Navier-Stokes/Darcy coupled problem

Passing to the limit with $\varepsilon = \delta t^2$ in the time scheme when $\delta t \to 0$

Theorem (Global solvability in time of the Navier-Stokes/Darcy coupled flow)

Let us consider any data $\boldsymbol{f} \in \boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega)^d)$ with $\boldsymbol{f}^p \in \boldsymbol{L}^2(0,T;\boldsymbol{H}^1(\Omega)^d)$, $\boldsymbol{f}_\Sigma \in \boldsymbol{L}^2(0,T;\boldsymbol{H}^{-1/2}(\Sigma)^d)$, $\boldsymbol{v}_0 \in \boldsymbol{V}$ and any $\boldsymbol{\mu} > 0$ with the above natural assumptions on $\boldsymbol{K}$, $\boldsymbol{\beta}_\Sigma$ and $\boldsymbol{\lambda}_\Sigma$.
Then, there exists at least a solution $(\boldsymbol{v},\boldsymbol{p})$ to the Navier-Stokes/Darcy problem for $\boldsymbol{d} \leq 3$ such that for any $\boldsymbol{T} > 0$:

i) $\boldsymbol{v} \in \boldsymbol{L}^2(0,T;\boldsymbol{V})$ *with* $\boldsymbol{v}^f \in \boldsymbol{L}^\infty(0,T;\boldsymbol{L}^2(\Omega_f)^d)$ *and* $\boldsymbol{v}^f$ *is weakly continuous from* $[0,T]$ *into* $\boldsymbol{L}^2(\Omega_f)^d$,

ii) $\boldsymbol{v}'^f = \partial_t \boldsymbol{v}^f \in \boldsymbol{L}^{4/d}(0,T;\boldsymbol{V}_f')$,

iii) $\boldsymbol{p} \in \boldsymbol{W}^{-1,\infty}(0,T;\boldsymbol{L}^2(\Omega))$ *with* $\boldsymbol{p}^p \in \boldsymbol{L}^2(0,T;\boldsymbol{H}^2(\Omega_p))$.

Moreover, any solution satisfies the energy inequality below for almost every $\boldsymbol{t} \in [0,T]$:

$$
\begin{aligned}
&\|\boldsymbol{v}(t)\|_{0,\Omega_f}^2 + \int_0^t \mu\,\|\nabla \boldsymbol{v}(\tau)\|_{0,\Omega_f}^2 \,\mathrm{d}\tau + \int_0^t \int_{\Omega_p} \mu\,(\boldsymbol{K}^{-1}(\tau)\boldsymbol{v}(\tau))\cdot\boldsymbol{v}(\tau)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau \\
&\qquad + \int_0^t \left(\int_\Sigma \frac{\mu}{\sqrt{K}}\,(\boldsymbol{\beta}_\Sigma(\tau)\,\boldsymbol{v}(\tau))\cdot\boldsymbol{v}(\tau)\,\mathrm{d}s\,\mathrm{d}\tau + \int_\Sigma |\boldsymbol{v}_\Sigma(\tau)|\,(\boldsymbol{\lambda}_\Sigma(\tau)\,\boldsymbol{v}(\tau))\cdot\boldsymbol{v}(\tau)\,\mathrm{d}s \right) \mathrm{d}\tau \\
&\qquad \leq \|\boldsymbol{v}_0\|_{0,\Omega_f}^2 + \int_0^t \left(\int_\Omega \boldsymbol{f}(\tau)\cdot\boldsymbol{v}(\tau)\,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{f}_\Sigma(\tau), \boldsymbol{v}(\tau)\rangle_{-1/2,\Sigma} \right) \mathrm{d}\tau.
\end{aligned}
$$

For the space dimension $\boldsymbol{d} = 2$, the solution $(\boldsymbol{v},\boldsymbol{p})$ is unique with $\boldsymbol{v}^f \in \mathscr{C}([0,T];\boldsymbol{L}^2(\Omega_f)^d)$ such that $\boldsymbol{v}'^f \in \boldsymbol{L}^2(0,T;\boldsymbol{V}_f')$ and the above inequality does actually become an equality of energy.

Some perspectives...

- Analysis of global well-posedness in time of the Stokes/Darcy-Brinkman and Stokes/Darcy fluid-porous coupled problems (for non-inertial flows) with different sets of jump interface conditions on $\boldsymbol{\Sigma_m}$ or $\boldsymbol{\Sigma_b}$:

  Ok whatever the size of data

  [PhA., ESAIM : Math. Model. Numer. Anal., 2018 and 2021 (submitted)]

- Global solvability in time of the Navier-Stokes/Darcy coupled problem with no restriction on the size of the data :

  N.B. All previous studies only prove the solvability
  either with small data (Reynolds number) for the steady problem
  or locally over a small time interval for the unsteady problem.

- Well-posedness analysis of the Navier-Stokes/Darcy-Forchheimer coupled problem : in progress

Thank you for your attention