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Regularity for optimal compliance problems with length penalization

Bohdan Bulanyi, Université de Paris, LJLL

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Statement of the problem

A spatial dimension $N \geq 2$ and an exponent $p \in (1, +\infty)$ are given. Let Ω be an open bounded set in $\mathbb{R}^N$ and let $f \in L^{q_0}(\Omega)$ with

$$q_0 = (p^*)' \text{ if } 1 < p < N, \quad q_0 > 1 \text{ if } p = N, \quad q_0 = 1, \text{ if } p > N$$

where $p^* = Np/(N-p)$ and $(1/p^*) + (1/(p^*)') = 1$.

We define the energy functional $E_{f,\Omega}$ over $W_0^{1,p}(\Omega)$ as follows

$$E_{f,\Omega}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega fudx.$$

For each closed proper subset Σ of $\overline{\Omega}$, $E_{f,\Omega}$ admits a unique minimizer $u_{f,\Omega,\Sigma}$ over $W_0^{1,p}(\Omega\backslash\Sigma)$, which is a unique solution to the Dirichlet problem

$$\begin{cases} -\Delta_p u &= f \text{ in } \Omega\backslash\Sigma \\ u &= 0 \text{ on } \Sigma \cup \partial\Omega, \end{cases}$$

which means that $u_{f,\Omega,\Sigma} \in W_0^{1,p}(\Omega\backslash\Sigma)$ and

$$\int_\Omega |\nabla u_{f,\Omega,\Sigma}|^{p-2}\nabla u_{f,\Omega,\Sigma}\nabla\varphi dx = \int_\Omega f\varphi dx \qquad \forall\varphi \in W_0^{1,p}(\Omega\backslash\Sigma)$$

For each closed proper subset Σ of $\overline{\Omega}$ we define the p-compliance functional at Σ by

$$C_{f,\Omega}(\Sigma) = -E_{f,\Omega}(u_{f,\Omega,\Sigma}) = \frac{1}{p'}\int_\Omega |\nabla u_{f,\Omega,\Sigma}|^p dx = \frac{1}{p'}\int_\Omega f u_{f,\Omega,\Sigma} dx$$

Physical interpretation, $N = 2$

- Ω can be interpreted as a membrane which is attached along $\Sigma \cup \partial\Omega$ to some fixed base and subjected to a given force f
- the space $W_0^{1,p}(\Omega \backslash \Sigma)$ consists of all kinematically admissible displacement fields
- the energy linear form

$$\varphi \mapsto \int_\Omega |\nabla u_{f,\Omega,\Sigma}|^{p-2} \nabla u_{f,\Omega,\Sigma} \nabla \varphi dx$$

  represents the work of the membrane at the equilibrium $u_{f,\Omega,\Sigma}$ and for an arbitrary displacement $\varphi \in W_0^{1,p}(\Omega \backslash \Sigma)$
- the value $E_{f,\Omega}(u_{f,\Omega,\Sigma})$ is the total potential energy at the equilibrium $u_{f,\Omega,\Sigma}$, which, according to the equilibrium equation, is equal to $-\frac{1}{p'} \int_\Omega f u_{f,\Omega,\Sigma} dx$

- the stiffness of the membrane is measured through the p-compliance functional which is equal to the product of the coefficient $\frac{1}{p'}$ and the work $\int_\Omega f u_{f,\Omega,\Sigma} dx$ performed by the force f.

Varying Σ in the class of all closed connected proper subsets of $\overline{\Omega}$, we want to find the maximal global stiffness of the membrane, provided that the weighted one-dimensional Hausdorff measure of Σ is taken into account.

Thus, we need to solve the following shape optimization problem, which we formulate in any spatial dimension $N \geq 2$.

Let $p \in (N-1, +\infty)$. Given $\lambda > 0$, find a set $\Sigma_{opt} \subset \overline{\Omega}$ minimizing the functional $\mathcal{F}_{\lambda,f,\Omega}$ defined by

$$\mathcal{F}_{\lambda,f,\Omega}(\Sigma) = C_{f,\Omega}(\Sigma) + \lambda \mathcal{H}^1(\Sigma)$$

among all sets Σ in the class $\mathcal{K}(\Omega)$ of all closed connected proper subsets of $\overline{\Omega}$.

Or simply solve the following shape optimization problem

$$(\mathcal{P}) \qquad \min_{\Sigma \in \mathcal{K}(\Omega)} \left(C_{f,\Omega}(\Sigma) + \lambda \mathcal{H}^1(\Sigma) \right)$$

Formulation in terms of stresses

Problem $(\mathcal{P})$ can be formulated in terms of stresses.

Expressing $-E_{f,\Omega}(u_{f,\Omega,\Sigma})$ in terms of the dual principle, we obtain the following dual formulation

$$(\mathcal{P}^*) \qquad \min_{\Sigma\in\mathcal{K}(\Omega)} \min_{\sigma\in S(\Sigma)} \frac{1}{p'}\int_\Omega |\sigma|^{p'}dx + \lambda\mathcal{H}^1(\Sigma)$$

of Problem $(\mathcal{P})$.

Here the minimization with respect to the stresses σ is taken over the set $S(\Sigma)$ of all statically admissible stresses fields, namely

$$S(\Sigma) = \{\sigma \in L^{p'}(\Omega;\mathbb{R}^N) : div(\sigma) + f = 0 \text{ in } \mathcal{D}'(\Omega\backslash\Sigma)\}.$$

Regularity in 2d of the minimizers of $(\mathcal{P})$, linear case

A. Chambolle, J. Lamboley, A. Lemenant, E. Stepanov, *Regularity for the optimal compliance problem with length penalization*, *SIAM J. Math. Anal., 2017*

- absence of closed loops
- local Ahlfors regularity inside Ω
- local $C^{1,\alpha}$ regularity inside Ω
- classification of blow-up limits inside Ω

Constrained problem

Problem (C). *Let* $p \in (N-1, +\infty)$*. Given* $L > 0$*, find a set* $\Sigma_{opt} \subset \overline{\Omega}$ *minimizing the p-compliance functional* $C_{f,\Omega}$ *among all sets* Σ *in the class of all closed connected subsets of* $\overline{\Omega}$ *satisfying the constraint* $0 < \mathcal{H}^1(\Sigma) \leq L$*.*

- G. Buttazzo, F. Santambrogio, *Asymptotical compliance optimization for connected networks*, *Netw. Heterog. Media, 2007*

-Behavior when $L \to \infty$;

-Γ-convergence to the average distance functional.

Let $\Omega \subset \mathbb{R}^N$ be open and bounded, $f \in L^{q_0}(\Omega)$ and $L > 0$.

$\mathcal{K}(\Omega)$ is endowed with the topology generated by the Hausdorff distance.

Then, as $p \to +\infty$, the functionals $(C_{f,\Omega}(\cdot))$ Γ-converge to the average distance functional given by

$$\mathcal{K}(\Omega) \ni \Sigma \mapsto \int_\Omega \operatorname{dist}(x, \Sigma \cup \partial\Omega) f(x) dx$$

The average distance problem

Thus, in some sense, the limit of Problem $(\mathcal{P})$ as $p \to +\infty$ corresponds to the minimization of the functional

$$\mathcal{K}(\Omega) \ni \Sigma \mapsto \int_{\Omega} \mathrm{dist}\,(x, \Sigma \cup \partial\Omega) f(x) dx + \lambda \mathcal{H}^1(\Sigma),$$

for which it is known that minimizers may not be C^1 regular

- D. Slepcev, *Counterexample to regularity in average-distance problem*, *Ann. Inst. H. Poincaré Anal. Non Linéaire, 2014*

Main results for minimizers of Problem $(\mathcal{P})$, $N \geq 2$

Several of our results will hold under some condition on the integrability of the source f. Namely we define

$$q_1 = \frac{Np}{Np - N + 1} \;\text{ if }\; 2 \leq p < +\infty, \qquad q_1 = \frac{2p}{3p-3} \;\text{ if }\; 1 < p < 2,$$

and we notice that $q_1 \geq q_0$.

The condition $f \in L^{q_1}(\Omega)$ for $p \in [2, +\infty)$ is natural, since q_1 in this case seems to be the right exponent which implies an estimate of the type

$$\int_{B_r(x_0)} |\nabla u|^p dx \leq Cr$$

for the solution u to the Dirichlet problem

$$-\Delta_p v = f \;\text{ in }\; B_r(x_0), \;\; v \in W_0^{1,p}(B_r(x_0))$$

Conventions: N will denote an integer greater than or equal to 2;

Ω, unless otherwise stated, will denote an open bounded set in $\mathbb{R}^N$

Local $C^{1,\alpha}$ regularity inside Ω

The main regularity result established is the following.

Theorem. *Let $p \in (N-1,+\infty)$, $f \in L^q(\Omega)$ with $q > q_1$. Then there exists a constant $\alpha \in (0,1)$ such that the following holds. Let Σ be a solution to Problem ($\mathcal{P}$). Then for $\mathcal{H}^1$-a.e. point $x \in \Sigma \cap \Omega$ one can find a radius $r_0 > 0$ depending on x such that $\Sigma \cap \overline{B}_{r_0}(x)$ is a $C^{1,\alpha}$ regular curve.*

Existence of nontrivial solutions

At least for some small enough values of λ, solutions to Problem $(\mathcal{P})$ are nontrivial.

Proposition. *Let $p \in (N-1, +\infty)$, $f \in L^{q_0}(\Omega)$, $f \neq 0$. Then there exists a number $\lambda_0 = \lambda_0(N, p, f, \Omega) > 0$ such that if* Problem $(\mathcal{P})$ *is defined for $\lambda \in (0, \lambda_0]$, then every solution to this problem has positive $\mathcal{H}^1$-measure. Moreover, if $p > N$, then the empty set will not be a solution to* Problem $(\mathcal{P})$.

Ahlfors regularity in 2d

A set $\Sigma \subset \mathbb{R}^N$ is said to be Ahlfors regular of dimension 1, if there exist constants $c > 0$, $r_0 > 0$ and $C > 0$ such that for every $r \in (0, r_0)$ and for every $x \in \Sigma$ the following holds

$$cr \leq \mathcal{H}^1(\Sigma \cap B_r(x)) \leq Cr$$

Ahlfors regularity in 2d

Theorem. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with locally Lipschitz boundary, $p \in (1, +\infty)$, and $f \in L^{\frac{2p}{2p-1}}(\Omega)$. Let Σ be a solution to* Problem $(\mathcal{P})$ *with diam$(\Sigma) > 0$. Then Σ is Ahlfors regular.*

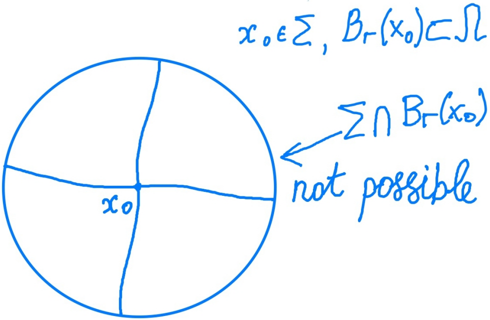
The exponent $q = \frac{2p}{2p-1}$ seems to be sharp

Absence of closed loops

Theorem. *Let* $p \in (N-1,+\infty)$ *and* $f \in L^q(\Omega)$ *with* $q > q_1$. *Then every solution* Σ *to* Problem $(\mathcal{P})$ *cannot contain closed loops (i.e., homeomorphic images of the unit circle* S^1*) and, therefore, topologically is a tree.*

Absence of quadruple points inside Ω

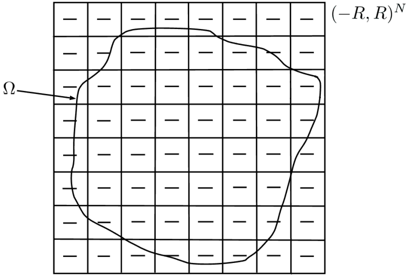
Proposition. *Let $p \in (N-1, +\infty)$ and $f \in L^q(\Omega)$ with $q > q_1$. Then every solution Σ to Problem $(\mathcal{P})$ cannot contain quadruple points in Ω.*



Importance of the connectedness assumption

Theorem. *Let* $\lambda > 0$, $p \in (N-1, +\infty)$, $f \in L^{q_0}(\Omega)$, $f \neq 0$. *Then the existence of minimizers for the functional* $C_{f,\Omega}(\cdot) + \lambda \mathcal{H}^1(\cdot)$ *over the class of all closed proper subsets of* $\overline{\Omega}$ *fails.*

Theorem. *Let* $L > 0$, $p \in (N-1, +\infty)$, $f \in L^{q_0}(\Omega)$, $f \neq 0$. *Then the existence of minimizers for the p-compliance functional* $C_{f,\Omega}$ *over the class* $\{\Sigma \subset \overline{\Omega} : \ \Sigma \text{ is closed}, \ 0 < \mathcal{H}^1(\Sigma) \leq L\}$ *fails.*



Some perspectives

- In 2d study the regularity of solutions to Problem $(\mathcal{P})$ up to the tip of an endpoint and up to the branching point for a triple point
- Is every solution Σ to Problem $(\mathcal{P})$ with $diam(\Sigma) > 0$ Ahlfors regular in any spatial dimension $N \geq 2$ for every $p \in (N-1, +\infty)$?
- Assume that Σ is a solution to Problem $(\mathcal{P})$ with $diam(\Sigma) > 0$, $S \subset \Sigma$ is such that any $x \in S$ is not a regular point, and $\Sigma \backslash S$ is locally $C^{1,\alpha}$ regular for some $\alpha \in (0,1)$. Is it true that $dim_H(S) < 1$?
- In codimension 2, can a solution to Problem $(\mathcal{P})$ have a knot?
- Classification of blow-up limits in 2d and for $p \neq 2$

Thank you for your attention!