Event-based control of the damped linear wave equation

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Context

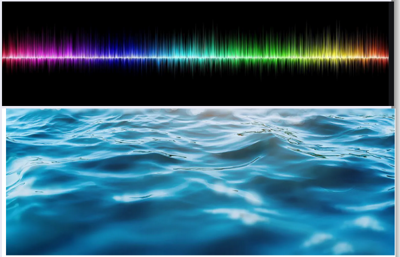
Wave equation : D'Alembert (1746), Euler (1756)

$$\partial_t^2 z(x,t) - \Delta z(x,t) = f(x,t)$$

Models propagation of waves (water, sound, seismic or light etc)

Wifi waves

Sound and water waves



Known results with internal feedback

Consider the wave equation in an open bounded domain Ω of $\mathbb{R}^N$:

(1)
$$\begin{cases} \partial_t^2 z(x,t) - \Delta z(x,t) = f(x,t) & \forall\ (x,t) \in \Omega \times (0,\infty), \\ z(x,t) = 0 & \forall\ (x,t) \in \partial\Omega \times (0,\infty), \\ z(x,0) = z_0(x) & \forall\ (x,t) \in \Omega, \\ \partial_t z(x,0) = z_1(x) & \forall\ x \in \Omega. \end{cases}$$

Energy: $E(t) = \frac{1}{2}(\|\partial_t z(t)\|^2_{L^2(\Omega)} + \|\nabla z(t)\|^2_{L^2(\Omega)})$

In [Chen(79'),Lions(88')] taking $\boxed{f(x,t) = -\alpha \partial_t z(x,t)}$ with $\alpha > 0$, and for every $(z_0, z_1) \in H_0^1(\Omega) \times L^2(\Omega)$

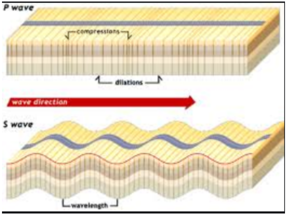
- $\rightsquigarrow$ $\exists!\, z \in C^0(\mathbb{R}^+; H_0^1(\Omega)) \cap C^1(\mathbb{R}^+; L^2(\Omega))$ solution to (1)
- $\rightsquigarrow$ Exponential stability : $\exists C > 0, \delta > 0,\ E(t) \leq CE(0)e^{-\delta t}$.

Control objective

We are interested in the **implementation of the feedback law** $f(x,t) = -\alpha\partial_t z(x,t_k)$ **on digital platforms**. A choice is the **periodic control** but one can design a **triggering strategy**, which determines the time instants when the control needs to be updated: it is the Event-Triggering Mechanism (ETM)

Control objective

Can we construct a ETM and maintain both stability and the well-posedness of the corresponding closed-loop system?



The proposed Event-triggering mechanism

Let us define the triggering law by

$$\begin{cases} t_0 &= 0, \\ t_{k+1} &= \inf \left\{ t \geq t_k \ \text{ such that } \| \underbrace{\partial_t z(x,t) - \partial_t z(x,t_k)}_{e_k(t)} \|^2_{L^2(\Omega)} > 2\gamma E(t) \right\} \end{cases}$$

where $\gamma > 0$ is a design parameter.

Consequence

$$\forall t \in [t_k, t_{k+1}), \ \|\partial_t z(x,t) - \partial_t z(x,t_k)\|^2_{L^2(\Omega)} \leq \ 2\gamma E(t).$$

The closed-loop system can then be described as follows:

(2)

$$\begin{cases} \partial_t^2 z(x,t) - \Delta z(x,t) = -\alpha \partial_t z(x,t_k), & \text{in } \Omega \times [t_k, t_{k+1}), k \in \mathbb{N} \\ z(x,t) = 0, & \text{on } \partial\Omega \times \mathbb{R}^+, \\ z(\cdot,0) = z_0, \partial_t z(\cdot,0) = z_1, & \text{in } \Omega. \end{cases}$$

Main result

Definition (Maximal time T of solution)

$$\begin{cases} T = +\infty & \textit{if } (t_k) \textit{ is a finite sequence,} \\ T = \limsup_{k \to +\infty} t_k & \textit{if not.} \end{cases}$$

Theorem (Exponential stability and avoidance of Zeno phenomenon)

1. $\forall (z_0, z_1) \in H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$,
there ***exists a unique solution to the closed-loop system under the designed ETM*** *such that*

$$z \in C^0([0,T); H^2(\Omega) \cap H^1_0(\Omega)) \cap C^1([0,T); H^1_0(\Omega))$$

and the Zeno behavior does not occur.

2. ***Under some matrix inequality condition****, the closed-loop system (system* (1) *with* $f(x,t) = -\alpha \partial_t z(x, t_k)$*) is exponentially stable*

Strategy to prove the well-posedness

Well-posedness [Baudouin,Marx, Tarbouriech(2019)]

- $\rightsquigarrow$ Based on Induction on every sample interval $[t_k, t_{k+1}]$,
- $\rightsquigarrow$ and well-posedness of the damped wave equation

Avoidance of Zeno behavior

- Consider $\forall t \in [t_k, t_{k+1}]\,, \varphi(t) = \frac{\|\partial_t z(x,t) - \partial_t z(x,t_k)\|^2_{L^2(\Omega)}}{2\gamma E(t)}$
- Prove that $\dot{\varphi}(t) \leq A + \frac{B}{\sqrt{E(t)}}$ and $E(0)e^{-2Ct} \leq E(t) \leq E(0)e^{2Ct}$
- Then $\forall k \in \mathbb{N}$, integrating on $[t_k, t_{k+1}]$ knowing that $\varphi(t_k) = 0$ and $\varphi(t_{k+1}) = 1$ we obtain: $1 \leq \left[A + \frac{Be^{CT}}{\sqrt{E(0)}}\right](t_{k+1} - t_k)$ showing there is **no accumulation points due to the update**. $\square$

Strategy to prove the exponential stability

Let us consider the following **Lyapunov function**:

$$V(t) = E(t) + \frac{\alpha\varepsilon}{2}\|z(t)\|^2_{L^2(\Omega)} + \varepsilon \int_\Omega z(x,t)\partial_t z(x,t)dx$$

1. **Step 1**: Equivalence of the energy and $V(t)$

$$E(t) \leq V(t) \leq \left(1 + \varepsilon C_\Omega + \varepsilon\alpha C_\Omega^2\right) E(t)$$

2. **Step 2**: Find the conditions on which for a desired decay rate δ

$$\dot{V}(t) + 2\delta V(t) \leq 0$$

This will imply the desired result : $E(t) \leq CE(0)e^{-2\delta t}$

Matrix inequality's condition to estimate $\dot{V}(t) + 2\delta V(t)$

- $\dot{V}(t) + 2\delta V(t) = \int_\Omega \psi^\top(x,t) M_1 \psi(x,t) dx$
  with $\psi = \begin{pmatrix} z \\ \partial_t z \\ e_k \\ \nabla z \end{pmatrix}$ and $M_1 = \begin{pmatrix} \alpha\varepsilon\delta & \delta\varepsilon & \frac{\alpha\varepsilon}{2} & 0 \\ \star & \varepsilon - \alpha + \delta & \frac{\alpha}{2} & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & \delta - \varepsilon \end{pmatrix}$.
- **Poincaré's inequality**:
  $\int_\Omega |z(t)|^2 dx \leq C_\Omega^2 \int_\Omega |\nabla z(t)|^2 dx \iff \int_\Omega \psi^\top(x,t) M_2 \psi(x,t) dx \geq 0$ with
  $M_2 = diag(-1, 0, C_\Omega^2, 0)$
- ETM : $\|e_k(t)\|_{L^2(\Omega)}^2 \leq 2\gamma E(t) \iff \int_\Omega \psi^\top M_3 \psi dx \geq 0$ with
  $M_3 = diag(0, \gamma, \gamma, -1)$

Matrix inequality's condition to estimate $\dot{V}(t) + 2\delta V(t)$

As a consequence

$$\dot{V}(t) + 2\delta V(t) = \int_\Omega \psi^\top M_1 \psi dx$$

is subject to $\int_\Omega \psi^\top(t) M_2 \psi(t) \geq 0$ and $\int_\Omega \psi^\top(t) M_3 \psi(t) \geq 0$.

S-procedure ensures the existence of

$\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ such that

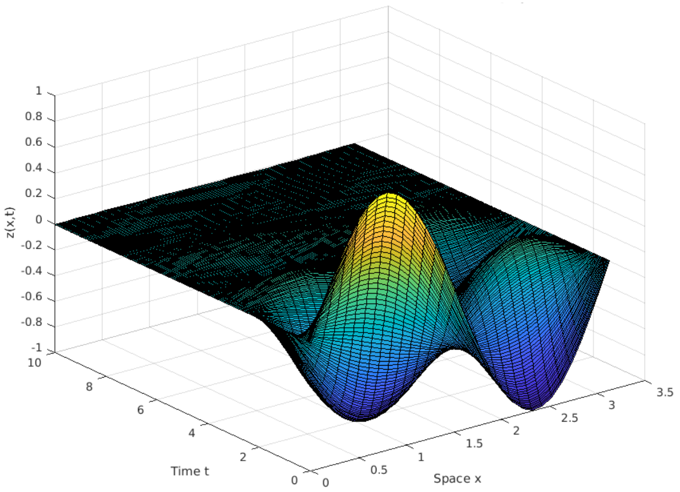
$$\int_\Omega \psi^\top (\underbrace{M_1 + \lambda_1 M_2 + \lambda_2 M_3}_{G}) \psi dx \leq 0$$

Feasibility of $G \prec 0$

$$G = \begin{pmatrix} -\lambda_1 + \alpha\varepsilon\delta & \delta\varepsilon & \frac{\alpha\varepsilon}{2} & 0 \\ \star & \phi_{22} & \frac{\alpha}{2} & 0 \\ \star & \star & -\lambda_2 & 0 \\ \star & \star & \star & \phi_{44} \end{pmatrix}$$

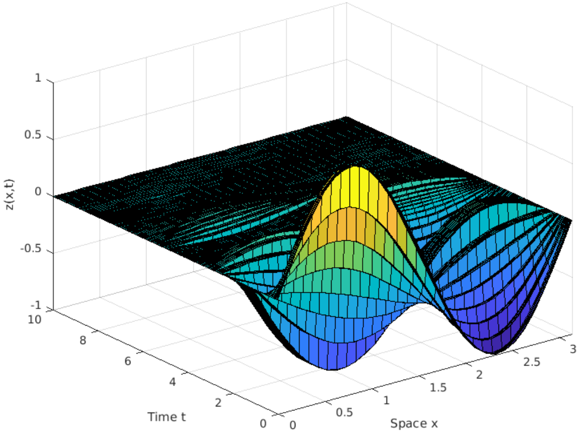
with $\phi_{22} = \varepsilon - \alpha + \delta + \bar{\gamma}$,
$\phi_{44} = \delta - \varepsilon + \lambda_1 C_\Omega^2 + \bar{\gamma}$
(Use Shur complement and Elimination lemma) □

Numerical Simulation 1D: $\partial_t^2 z - \partial_x^2 z = -\alpha \partial_t z(t)$



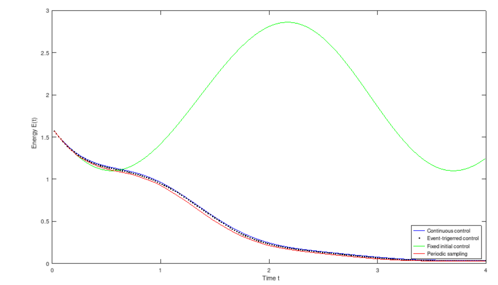
$\lambda_1 = 0.1, \lambda_2 = 1, \gamma = 0.02, \delta = 0.25$ and $\varepsilon = 0.8$ with $\alpha = 1$.

Numerical Simulation 1D: $\partial_t^2 z - \partial_x^2 z = -\alpha \partial_t z(t_k)$

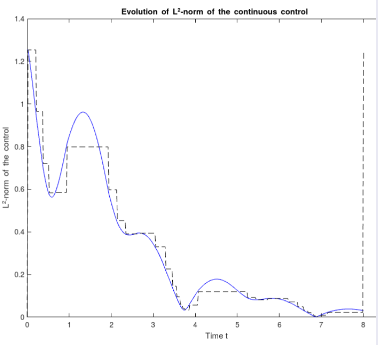


Evolution of the energy and the control

Evolution of the energy $E(t)$



Evolution of $\|\alpha \partial_t z\|_{L^2(\Omega)}$



Conclusion and Future works

We present a matrix inequality approach for the **exponential stabilization** of the damped linear **wave equation** under an **event-triggering mechanism**.

Future works

1. What about the localized damping coefficient α?
2. What about the case of the boundary control ?
3. It would be relevant to study other classes of PDEs like the Schrödinger equation

Thank you for your attention!

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