A multigrid solver for the M_1 model for radiative transfer

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10 ième Biennale Française des Mathématiques Appliquées et Industrielles

MAISON DE LA SIMULATION

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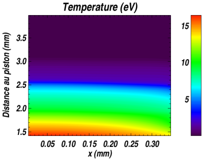
Radiative transfer

Many physical situations

Radiative transfer



[González, 2006]

Many physical situations

- Laser-matter interactions

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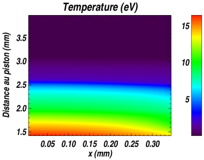
- Astrophysics: HII region, . . .



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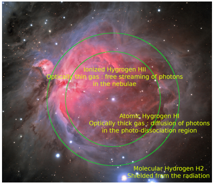
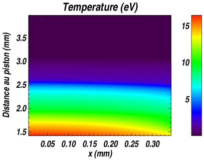


Many physical situations

- Laser-matter interactions
- Astrophysics: HII region, . . .
- Atmospheric physics



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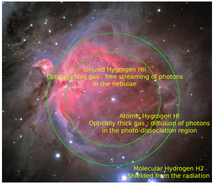
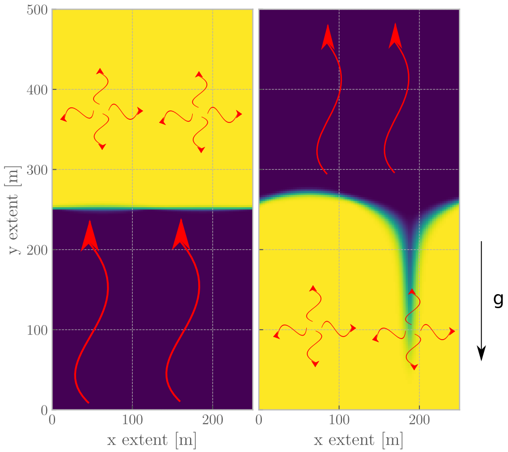
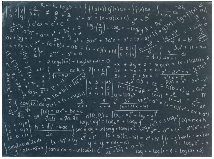


[Tremblin et al., prep]

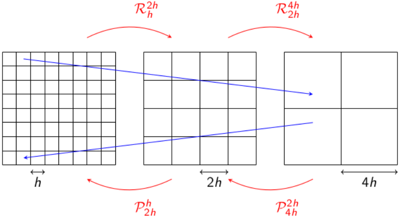
Radiative transfer in atmospheric physics

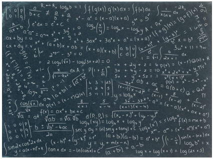


[Tremblin et al., prep] : opacity jump interface



1. M_1 model
2. A geometric multigrid solver
3. Conclusion





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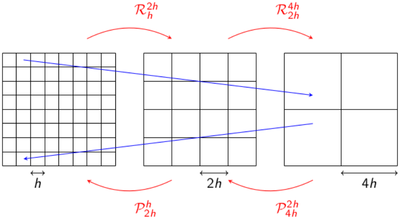

Moment method

- Grey radiative transfer equation without scattering [1]

$$\left(\frac{1}{c}\partial_t + \mathbf{n}\cdot\nabla\right) I(\mathbf{x}, t, \mathbf{n}) = \sigma B(\mathbf{x}, t) - \sigma I(\mathbf{x}, t, \mathbf{n})$$

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- Admissible states: $E_r > 0$ and $\frac{||\mathbf{F}_r||}{cE_r} \leq 1$

[1] [Mihalas and Mihalas, 1984]

Closure relation

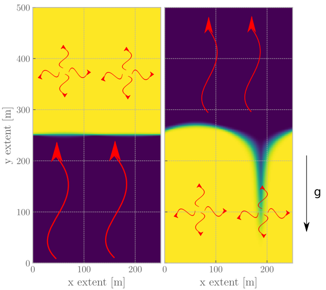
M_1 model closure relation [2]

$$\mathbb{P}_r = \mathbb{D} E_r$$

where $\mathbb{D}$ is the Eddington tensor

$$\mathbb{D} = \frac{1-\chi}{2}\mathbb{I} + \frac{3\chi - 1}{2}\mathbf{n} \otimes \mathbf{n}$$

- Free-streaming regime: $\chi(E_r, \mathbf{F}_r) = 1$, $\mathbb{D} = \mathbf{n} \otimes \mathbf{n}$
- Diffusive limit: $\chi(E_r, \mathbf{F}_r) = \frac{1}{3}$, $\mathbb{D} = \frac{1}{3}\mathbb{I}$



[2] [Dubroca and Feugeas, 1999]

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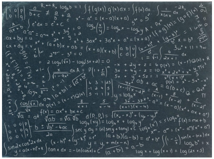

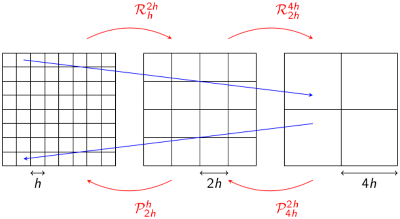

Implicit solver

- Speed of light
  $c = 3 \times 10^5\,\mathrm{km\,s^{-1}}$
- Speed of sound
  $v = 10\,\mathrm{km\,s^{-1}}$
- $\frac{c}{v} = 3 \times 10^4$
- Implicit scheme with time step
  $\Delta t = 3 \times 10^4 \frac{h}{c}$

Newton method

- Compute the Jacobian

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- Solve linear system

Newton method

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- Solve linear system
- Does not preserve the admissible states $E_r > 0$ and $\frac{||\mathbf{F}_r||}{cE_r} \leq 1$

Jacobi method

$$E_i^{n+1}\left(1+c\frac{\Delta t}{h}\right)=E_i^n$$
$$+\frac{\Delta t}{2h}\left(cE_{i+1}^{n+1}-F_{i+1}^{n+1}\right)+\frac{\Delta t}{2h}\left(cE_{i-1}^{n+1}+F_{i-1}^{n+1}\right)$$
$$F_i^{n+1}\left(1+c\frac{\Delta t}{h}\right)=F_i^n$$
$$+\frac{c\Delta t}{2h}\left(F_{i+1}^{n+1}-cP_{i+1}^{n+1}\right)+\frac{c\Delta t}{2h}\left(F_{i-1}^{n+1}+cP_{i-1}^{n+1}\right)$$

[Pichard, 2016]

Jacobi method

$$E_i^{n+1,k+1}\left(1+c\frac{\Delta t}{h}\right)=E_i^n$$
$$+\frac{\Delta t}{2h}\left(cE_{i+1}^{n+1,k}-F_{i+1}^{n+1,k}\right)+\frac{\Delta t}{2h}\left(cE_{i-1}^{n+1,k}+F_{i-1}^{n+1,k}\right)$$
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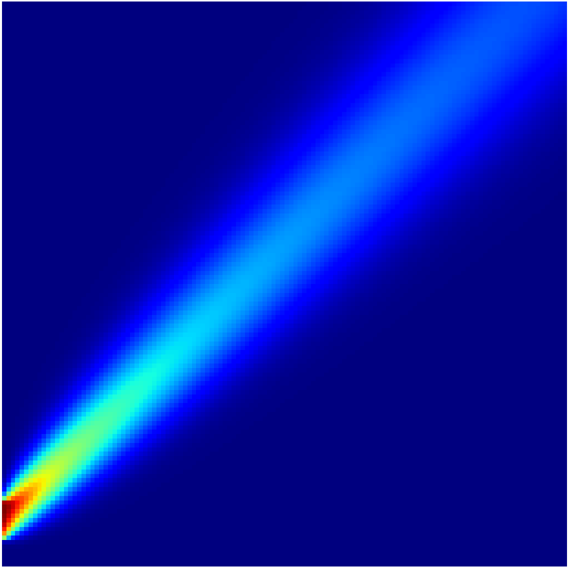
[Pichard, 2016]

Beam test



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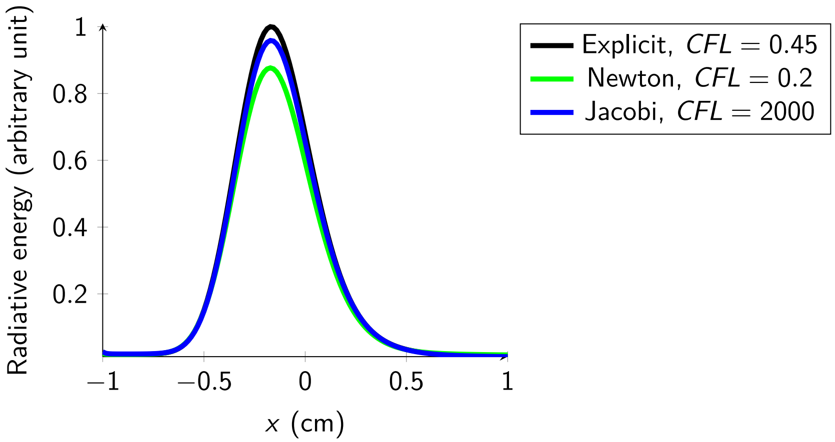

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Radiative energy (arbitrary unit)
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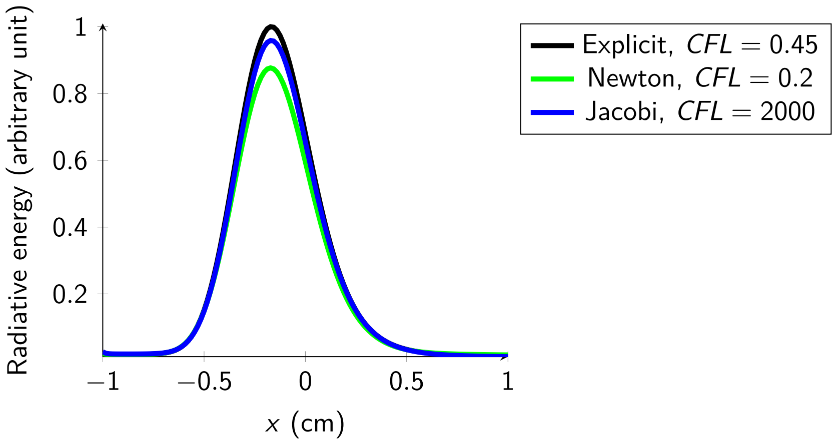
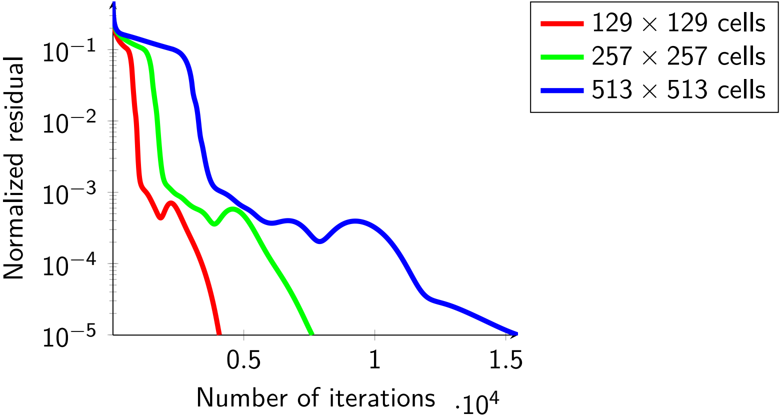


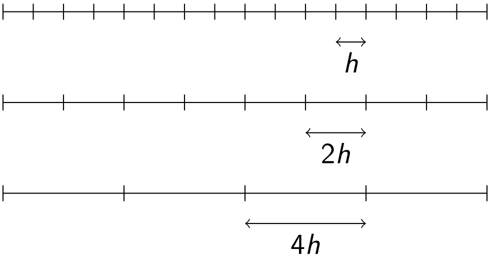
Jacobi three times faster than explicit solver

Convergence

$v_i = (E_i, F_i)$
Solve $\mathcal{A}(\mathbf{v}) = \mathbf{f}$
Residual: $||\mathbf{f} - \mathcal{A}(\mathbf{v})||$



Geometric multigrid (GMG)



[Briggs et al., 2000]

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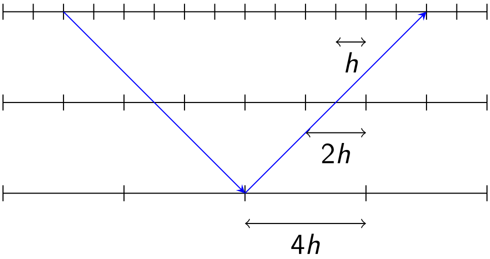

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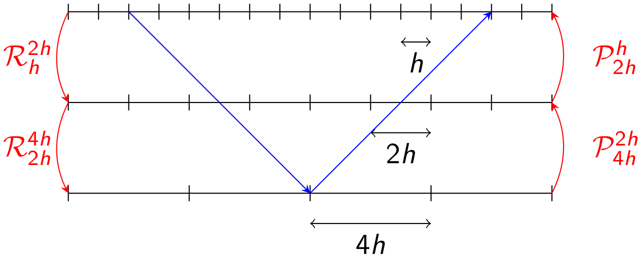

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Full Approximation Scheme (FAS)

- Pre-smoother: relax a few times $\mathcal{A}^h\left(\mathbf{v}^h\right) = \mathbf{f}^h$.

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- Post-smoother: relax a few times $\mathcal{A}^h\left(\mathbf{v}^h\right) = \mathbf{f}^h$.

[Briggs et al., 2000]

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Pseudo-time equation

Instead of solving

$$\mathcal{A}^{2h}\left(\mathbf{u}^{2h}\right) = \mathbf{r}^{2h} + \mathcal{A}^{2h}\left(\mathbf{v}^{2h}\right),$$

[Kifonidis and Müller, 2012]

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$$\frac{\mathrm{d}\mathbf{u}^{2h}}{\mathrm{d}\tau} + \mathcal{A}^{2h}\left(\mathbf{u}^{2h}\right) = \mathbf{r}^{2h} + \mathcal{A}^{2h}\left(\mathbf{v}^{2h}\right),$$

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as

$$\frac{\widetilde{\mathbf{u}^{2h}} - \left(\mathbf{u}^{2h}\right)^{m}}{\Delta\tau} = \mathbf{r}^{2h} + \mathcal{A}^{2h}\left(\mathbf{v}^{2h}\right)$$

$$\frac{\left(\mathbf{u}^{2h}\right)^{m+1} - \widetilde{\mathbf{u}^{2h}}}{\Delta\tau} + \mathcal{A}^{2h}\left(\left(\mathbf{u}^{2h}\right)^{m+1}\right) = 0$$

[Kifonidis and Müller, 2012]

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$$\left(\mathbf{u}^{2h}\right)^{m+1} + \Delta\tau\mathcal{A}^{2h}\left(\left(\mathbf{u}^{2h}\right)^{m+1}\right) = \widetilde{\mathbf{u}^{2h}}$$

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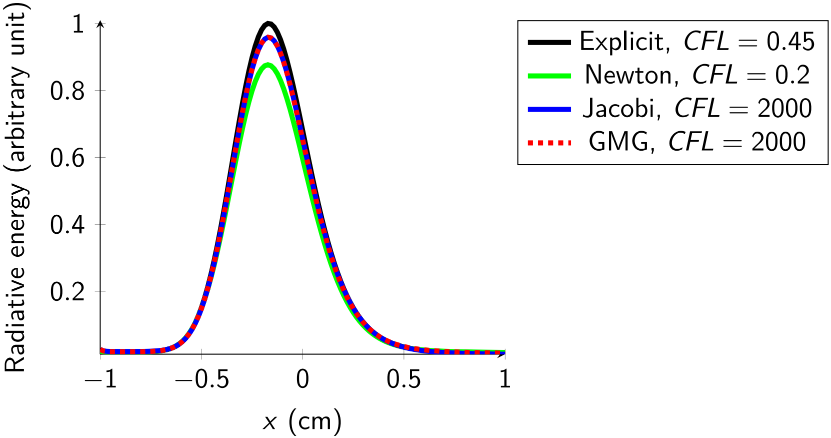

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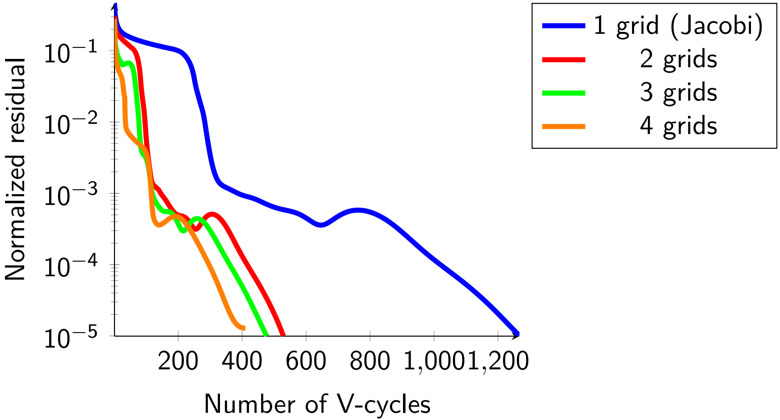
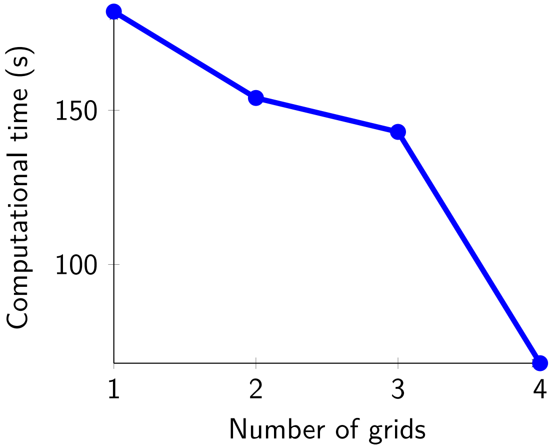
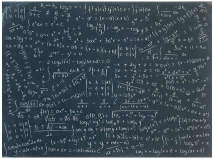


Computational time





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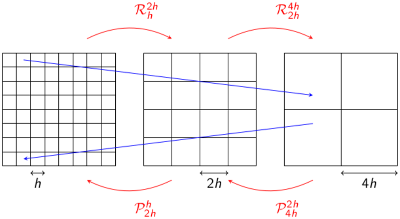
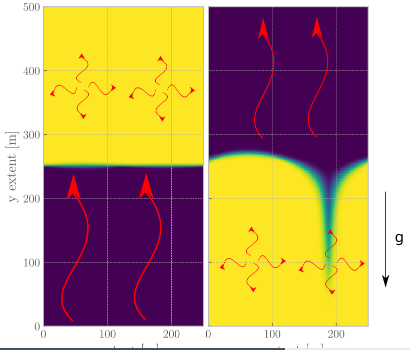


Conclusion and perspectives

- Conclusion
  - Implicit scheme with Jacobi method
  - Geometric multigrid
- Perspectives
  - Source terms
  - Coupling to hydrodynamics for physical applications



Prolongation operator

$$
\left(\mathcal{P}_{2h}^{h}\left(\mathbf{v}^{2h}\right)\right)_{i^h,j^h} = \begin{cases}
\mathbf{v}_{i^{2h},j^{2h}}^{2h} \text{ if } i^h \text{ is even and } j^h \text{ is even,} \\
\frac{1}{2}\left(\mathbf{v}_{i^{2h},j^{2h}}^{2h} + \mathbf{v}_{i^{2h}+1,j^{2h}}^{2h}\right) \text{ if } i^h \text{ is odd and } j^h \text{ is even,} \\
\frac{1}{2}\left(\mathbf{v}_{i^{2h},j^{2h}}^{2h} + \mathbf{v}_{i^{2h},j+1^{2h}}^{2h}\right) \text{ if } i^h \text{ is even and } j^h \text{ is odd,} \\
\frac{1}{4}\left(\mathbf{v}_{i^{2h},j^{2h}}^{2h} + \mathbf{v}_{i+1^{2h},j^{2h}}^{2h} + \mathbf{v}_{i^{2h},j^{2h}+1}^{2h} + \mathbf{v}_{i^{2h}+1,j^{2h}+1}^{2h}\right) \\
\quad \text{if } i^{2h} \text{ is odd and } j^{2h} \text{ is odd.}
\end{cases}
$$

[Strang, 2006]

Restriction operator

$$\left(\mathcal{R}_h^{2h}\left(\mathbf{v}^h\right)\right)_{i^{2h},j^{2h}} = \frac{1}{16}\mathbf{v}^h_{i^h-1,j^h-1} + \frac{1}{8}\mathbf{v}^h_{i^h-1,j^h} + \frac{1}{16}\mathbf{v}^h_{i^h-1,j^h+1} + \frac{1}{8}\mathbf{v}^h_{i^h,j^h-1} + \frac{1}{4}\mathbf{v}^h_{i^h,j^h} + \frac{1}{8}\mathbf{v}^h_{i^h,j^h+1} + \frac{1}{16}\mathbf{v}^h_{i^h+1,j^h-1} + \frac{1}{8}\mathbf{v}^h_{i^h+1,j^h} + \frac{1}{16}\mathbf{v}^h_{i^h+1,j^h+1}.$$

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