A fixed point approach to Clausius-Mossotti formulas

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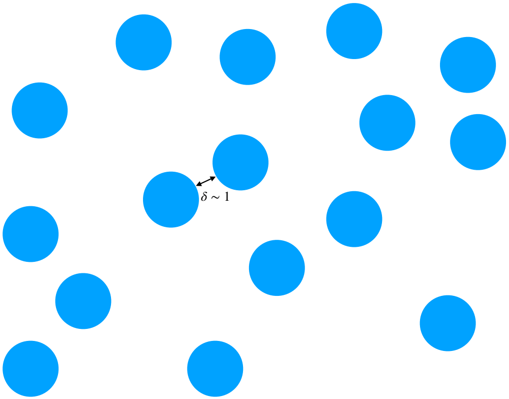
Two-phase material



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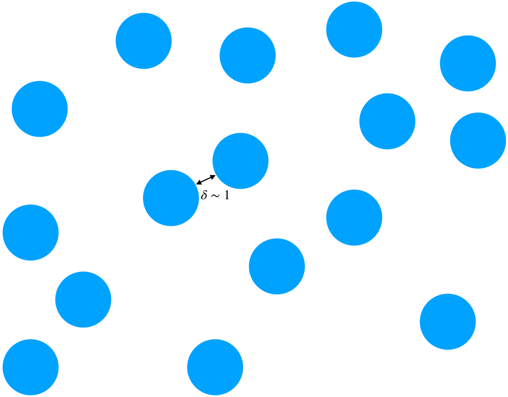

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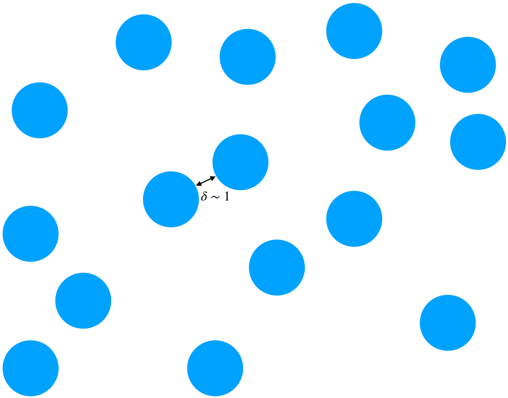

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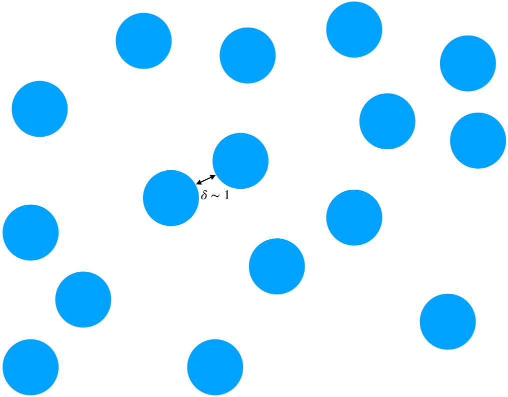

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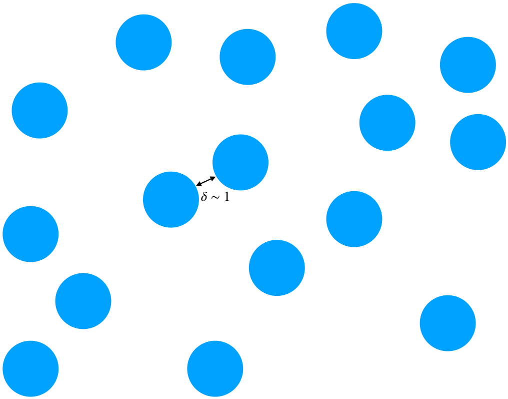

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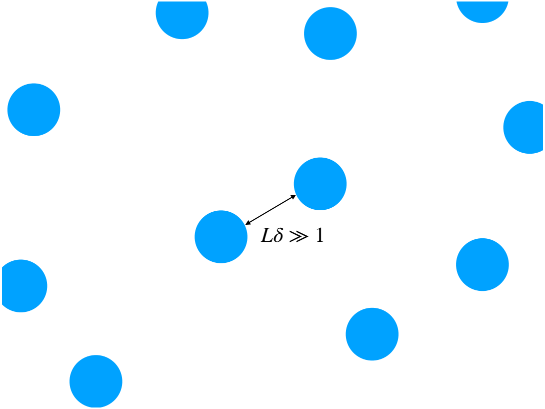

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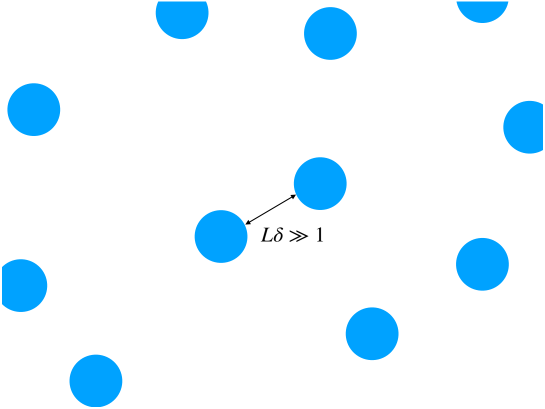

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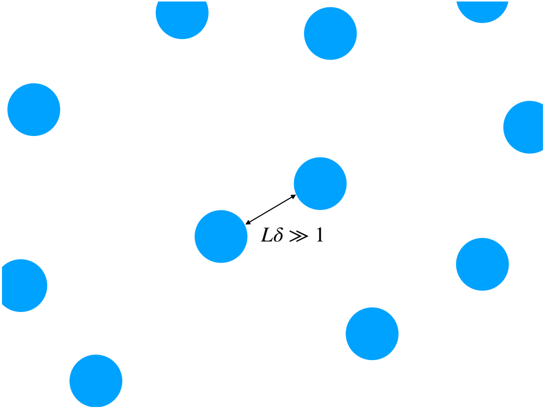

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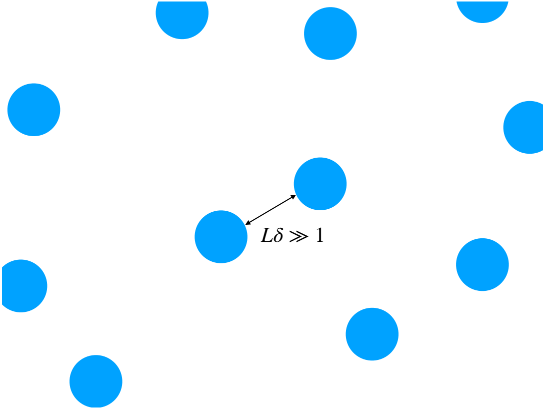

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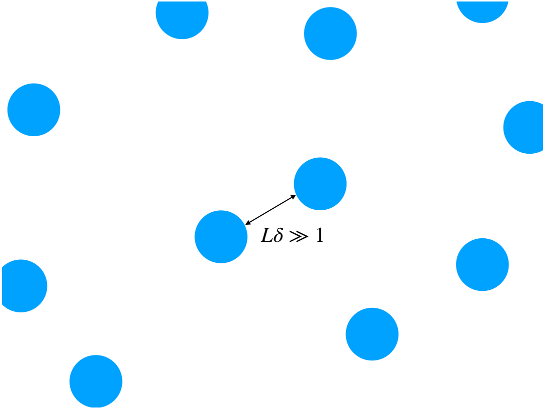

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Main question

What can we say about $L \mapsto \overline{a}^L(\mathcal{P})$?

Outline

- Main result
- Fixed point approach
- Extensions

Theorem (P. '21)

For $\mathcal{P}$ hardcore stationnary ergodic, with arbitrarily small α-mixing then $L^{-1} \mapsto \overline{a}^L$ can be Taylor expanded at 0 (to any order).

There exists, $(\mathcal{A}^{(i)})_{i\in\mathbb{N}} \in \left(\mathbb{R}^{d\times d}\right)^{\mathbb{N}}$ s.t $\forall n \geq 0$,

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Theorem (P. 21')

For periodic $\mathcal{P}$, $L^{-1} \mapsto \overline{a}^L$ analytic at 0.

$$\overline{a}^L e = \alpha e + L^{-d} \int_{B_1} \beta \left(\mathrm{Id} - L^{-d} \mathcal{B}^\circ K^L\right)^{-1} (\nabla\varphi^\circ + e)$$

with $L^{-1} \mapsto K^L$ analytic.

0^{th} order reformulation

$$\overline{a}^{L} e = \alpha e + L^{-d} \int_{B_1} \beta \mathbb{E}^{\circ} \left[(\nabla \varphi + e)(L\mathcal{P}^{\circ}, x) \right] \, \mathrm{d}x$$

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Indeed,

$$\overline{a}^L e = \mathbb{E}\left[\left(\alpha + \sum_{x \in \mathcal{P}} \beta \mathbb{1}_{B_1}(\cdot - Lx) \right) (\nabla \varphi + e)\,(L\mathcal{P}, \cdot) \right](0)$$

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&= \alpha e + \int_{\mathbb{R}^d} \beta \mathbb{1}_{B_1}(-Lx) \mathbb{E}^\circ \left[(\nabla \varphi + e) (L\mathcal{P}^\circ, -Lx) \right] \theta \, \mathrm{d}x
\end{aligned}$$

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Ref : Rayleigh-Maxwell [JKO], Berdichevsky, Anantharaman-Le Bris, Duerinckx-Gloria, . . .

Formal computations

- PDE for the $\varphi(L\mathcal{P}^\circ) - \varphi^\circ$

$$-\nabla \cdot a^\circ \nabla \left(\varphi(L\mathcal{P}^\circ) - \varphi^\circ\right) = \nabla \cdot \sum_{\substack{x \in \mathcal{P}^\circ \\ x \neq 0}} \beta \mathbb{1}_{B_1(Lx)} \left(\nabla \varphi(L\mathcal{P}^\circ) + e\right)$$

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α-mixing comes to the rescue

Definition (Algebraic α-mixing)

$\exists\ \alpha > 0$ s.t for all $U, V \subset \mathbb{R}^d$, $A \in \sigma(\mathcal{P}_{|U})$, $B \in \sigma(\mathcal{P}_{|V})$,

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- Optimization strategy :

$$\overline{a}_T^L = \mathrm{Id} + \sum_{i=d}^{n} \mathcal{A}_T^{(i)} L^{-i} + O\left(\ln(T)^{n+1} L^{-(n+1)}\right)$$

$$|\overline{a}_T^L - \overline{a}^L| \lesssim T^{-\beta} \Longrightarrow \forall i,\ \mathcal{A}_T^{(i)} \xrightarrow[T\infty]{} \mathcal{A}^{(i)} + \text{Taylor } \overline{a}^L$$

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Thanks for your attention !