Optimal Sampling and Weighted Least-Squares

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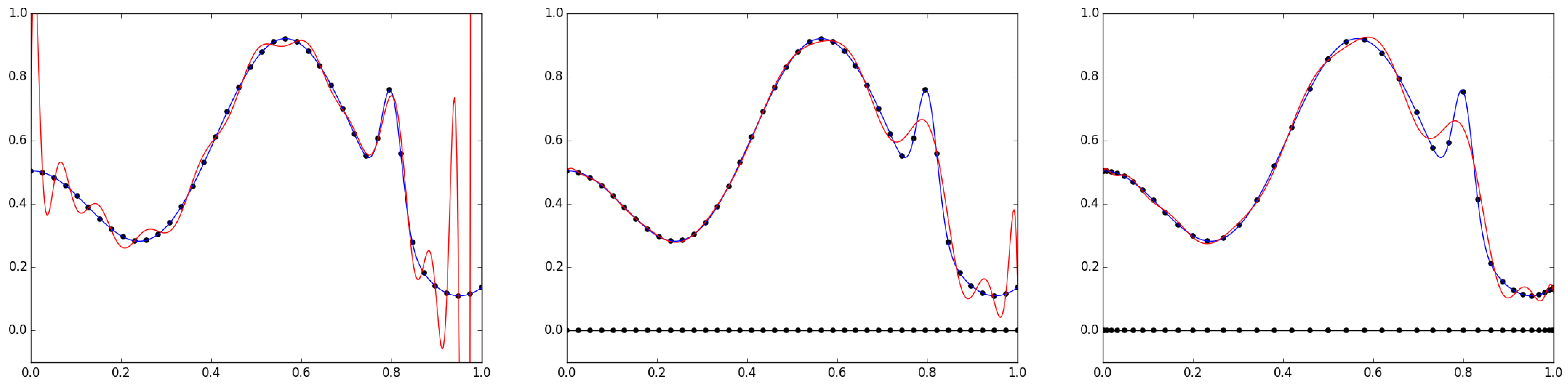
Introduction

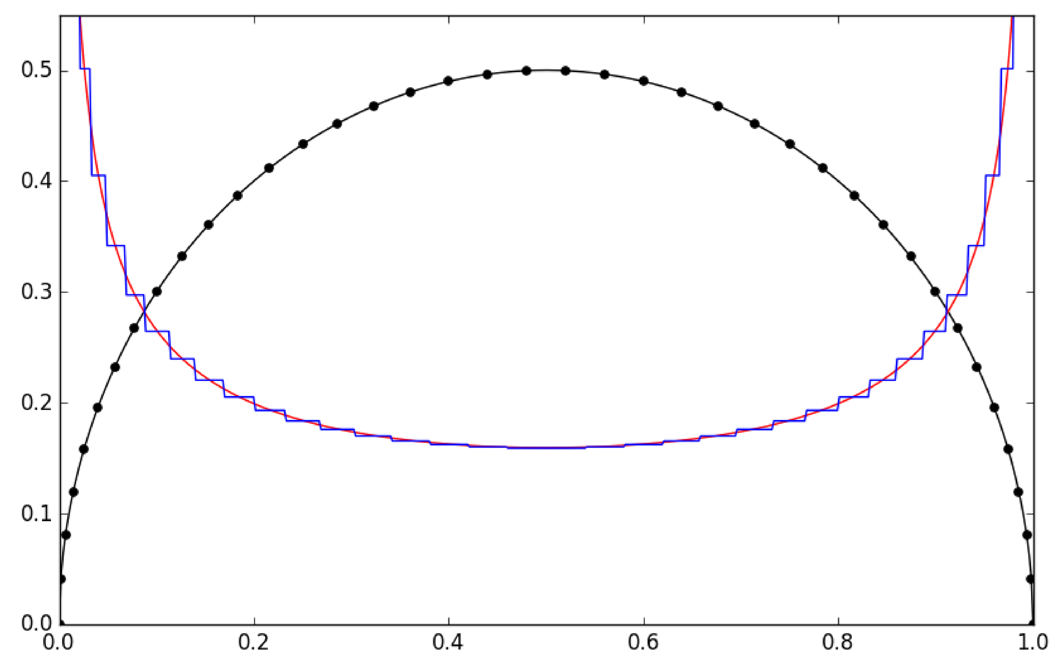


We want to approximate an unknown function $\mathbf{u} \in L^2(D)$, with $D \subset \mathbb{R}^d$ a compact domain, by some $\tilde{\mathbf{u}} \in V_n$, with $V_n \subset L^2(D)$ a fixed space of dimension n, based on evaluations of u at m chosen points $\mathbf{x}^1, \dots, \mathbf{x}^m \in D$, with m of the order of n.

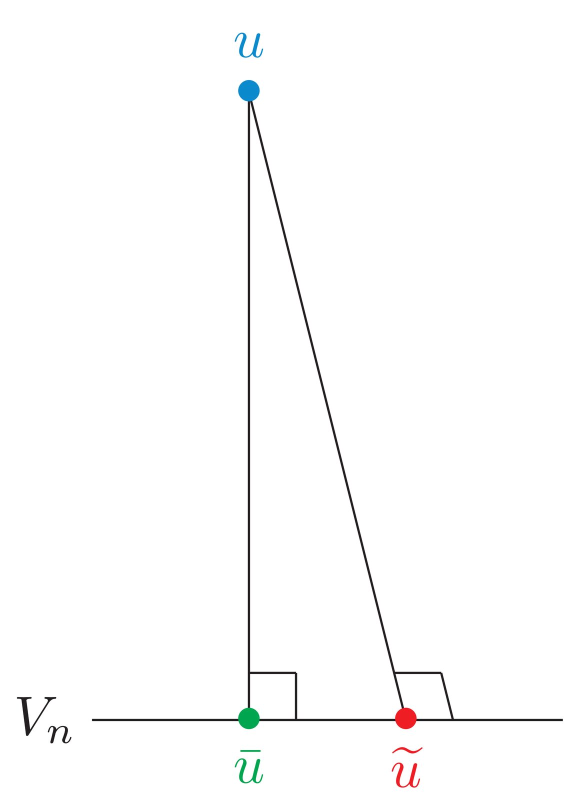
Define $\bar{\mathbf{u}} = P_{L^2} u \in V_n$, it reaches the optimum $\|u - \bar{u}\|_{L^2} = \min_{v \in V_n} \|u - v\|_{L^2}$

Goal: Given D and $V_n \subset L^2(D)$, choose m **random** points $x^1, \dots, x^m \in D$ such that for any $u \in L^2(D)$, the approximation $\tilde{u} \in V_n$ satisfies

$$\mathbb{E}\|u - \tilde{u}\|^2_{L^2(D)} \leq C\|u - \bar{u}\|^2_{L^2(D)}$$



Weighted Least-Squares



Define the discrete ℓ^2 semi-norm

$$\|v\|^2_{\mathbf{x}} := \frac{1}{m}\sum_{i=1}^m w(x^i)|v(x^i)|^2$$

Weighted least-squares approximation

$$\tilde{u} := \underset{v \in V_n}{\operatorname{argmin}} \|u - v\|^2_{\mathbf{x}} = P_{\mathbf{x}} u = \sum_{k=1}^n c_k \phi_k,$$

where $(\phi_1, \dots, \phi_n)$ is an orthonormal basis of V_n and c is the solution to $Gc = B$ with

$$B_j = \langle \phi_j, u \rangle_{\mathbf{x}} \text{ and } G_{j,k} = \langle \phi_j, \phi_k \rangle_{\mathbf{x}} = \sum_{i=1}^m a_i a_i^*, \; a_i = \left(\sqrt{\frac{w(x^i)}{m}} \phi_j(x^i) \right)_{j \leq n}$$

For any $v = \sum_{k=1}^n c_k \phi_k$, we have $\|v\|^2_{L^2} = c^* c$ and $\|v\|^2_{\mathbf{x}} = c^* G c$

Concentration Inequalities

Lemma: If $\mathbb{E}(\|v\|^2_{\mathbf{x}}) = \|v\|^2_{L^2}$ for $v \in L^2(D)$ and $\|v\|^2_{L^2} \leq C\|v\|^2_{\mathbf{x}}$ for $v \in V_n$, then

$$\mathbb{E}(\|u - \tilde{u}\|^2_{L^2}) \leq (1 + C)\|u - \bar{u}\|^2_{L^2}$$

$\Rightarrow$ Draw i.i.d points (x^i) according to $d\sigma(x) = \frac{1}{w(x)}dx$

$\Rightarrow$ Minimise $|a_i|^2 = \frac{w(x^i)}{m}\sum_{j=1}^n |\phi_j(x^i)|^2$ by taking

$$\frac{1}{w(x)} = k_n(x) := \frac{1}{n}\sum_{j=1}^n |\phi_j(x^i)|^2$$

Theorem (Ahlswede and Winter 2002, Tropp 2012)**:**

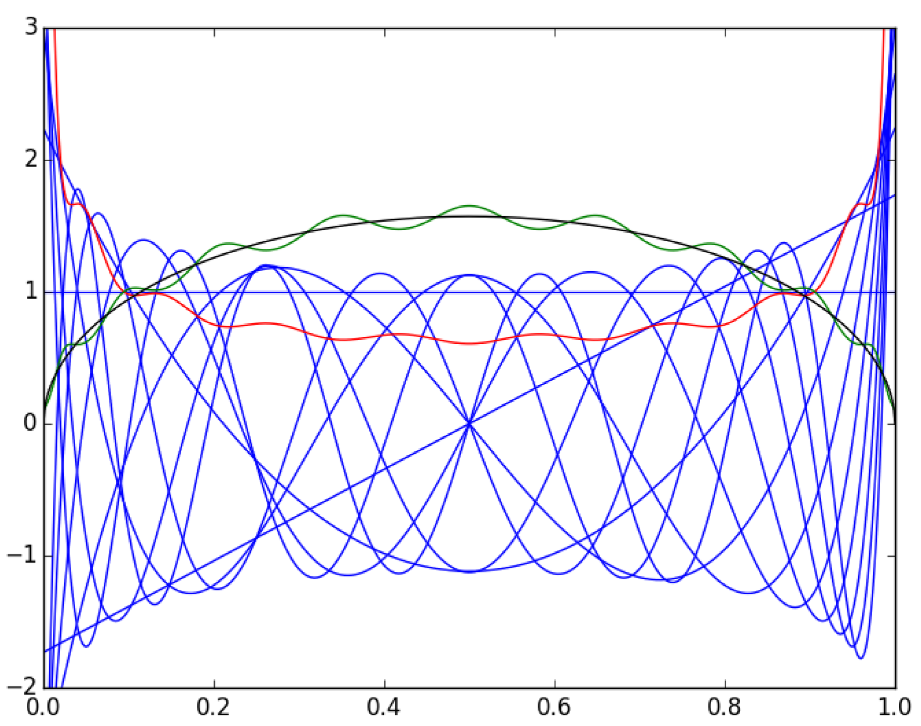
$$\mathbb{P}(\|G - I\| > 1/2) \leq 2ne^{-\frac{m}{10n}}$$

Corollary: If $m \geq 10n\log(2n/\varepsilon)$, then $\frac{1}{2}I \leq G \leq \frac{3}{2}I$ with probability at least $1 - \varepsilon$

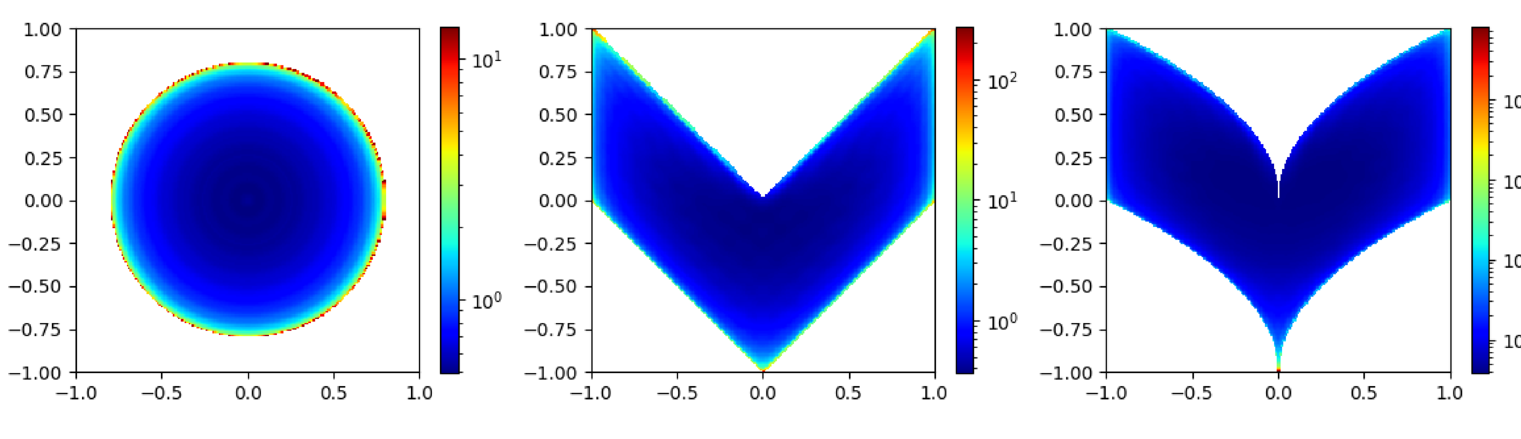
Theorem (Cohen and Migliorati, 2017): If $m \geq 10n\log(2n/\varepsilon)$, then

$$\mathbb{E}(\|u - \tilde{u}\|^2_{L^2}) \leq 3\,\|u - \bar{u}\|^2_{L^2} + \mathcal{O}(\varepsilon)$$

Christoffel function



$$k_n(x) = \frac{1}{n}\sum_{j=1}^n |\phi_j(x^i)|^2 = \max_{v \in V_n} \frac{|v(x)|^2}{\|v\|^2_{L^2}}$$



Sample Reduction

Theorem (Marcus, Spielman and Srivastava, 2015)**:** Let $r \geq 2$ and $a_1, \dots, a_m \in \mathbb{C}^n$ such that $|a_i|^2 \leq \delta$ and $\sum_{i=1}^m a_i a_i^* = I$.
Then there exists a partition $S_1 \sqcup \dots \sqcup S_r$ of $\{1, \dots, m\}$ such that

$$\sum_{i \in S_j} a_i a_i^* \leq \frac{(1 + \sqrt{r\delta})^2}{r} I, \quad j = 1, \dots, r$$

Theorem (Cohen and Dolbeault, 2021)**:** There exists a random sampling $S = \{x^1, \dots, x^{\bar{m}}\}$ with $\bar{m} \leq Cn$ and a linear reconstruction $(u(x^i)) \mapsto \tilde{u}$ such that

$$\mathbb{E}\|u - \tilde{u}\|^2_{L^2} \leq C'\|u - \bar{u}\|^2_{L^2}$$

Drawbacks:

- C and C' are inexploitably large
- exponential complexity for constructing the sample