A Grad closure for low-temperature plasmas

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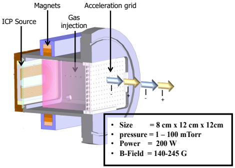
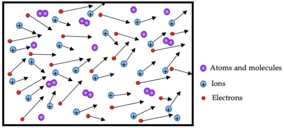
Outline of the talk

Motivation of this work

The motivation of this work is PEGASES, a gridded ion thruster that operates with a low-temperature plasma.





$$\boxed{\begin{array}{l} n_g \gg n_e, n_i \\ T_e \gg T_g, T_i \end{array}}$$

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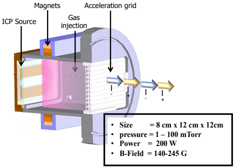

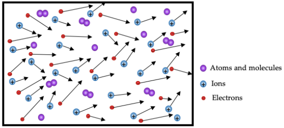

$n_{\mathfrak{g}} \gg n_{\mathfrak{e}}, n_{\mathfrak{i}}$

$T_{\mathfrak{e}} \gg T_{\mathfrak{g}}, T_{\mathfrak{i}}$

We consider a kinetic model for the electrons in an electropositive atomic plasma, consisting of electrons ($\mathfrak{e}$), a single species of positive ions ($\mathfrak{i}$), and neutral gas ($\mathfrak{g}$). The kinetic equation for the electrons reads:

$$\frac{\partial f_{\mathfrak{e}}}{\partial t} + \mathbf{v} \cdot \nabla f_{\mathfrak{e}} + \frac{e \nabla \phi}{m_{\mathfrak{e}}} \cdot \nabla_{\mathbf{v}} f_{\mathfrak{e}} = \left. \frac{\delta f_{\mathfrak{e}}}{\delta t} \right|_c \qquad (1)$$

We consider the following collisional processes

$$\left.\frac{\delta f_{\mathfrak{e}}}{\delta t}\right|_{c} = \left.\frac{\delta f_{\mathfrak{e}}}{\delta t}\right|_{\mathfrak{eg}}^{(elast.)} + \left.\frac{\delta f_{\mathfrak{e}}}{\delta t}\right|_{\mathfrak{eg}}^{(inelast.)} + \left.\frac{\delta f_{\mathfrak{e}}}{\delta t}\right|_{\mathfrak{ee}} + \left.\frac{\delta f_{\mathfrak{e}}}{\delta t}\right|_{\mathfrak{ei}}.$$

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electron-gas elastic collisions

- Boltzmann operator

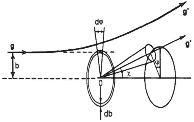
$$\left.\frac{\delta f_{\mathfrak{e}}}{\delta t}\right|_{\mathfrak{e}\,\mathfrak{g}}^{(Boltz)} = \int\int \left(f'_{\mathfrak{e}} f'_{\mathfrak{g}} - f_{\mathfrak{e}} f_{\mathfrak{g}}\right) g\sigma d\Omega d\mathbf{v}_{\mathfrak{g}} .$$

where:

$g = |\mathbf{v}_{\mathfrak{g}} - \mathbf{v}_{\mathfrak{e}}|$ is the relative velocity

$\sigma(g, \chi)$ is the differential scattering cross-section

$d\Omega = \sin\chi d\chi d\varphi$ is the differencial solid angle



Electron collisions: Collisional operators

We consider the following collisional processes

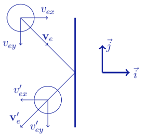
$$\left.\frac{\delta f_{\mathfrak{e}}}{\delta t}\right|_{c} = \left.\frac{\delta f_{\mathfrak{e}}}{\delta t}\right|_{\mathfrak{e}\mathfrak{g}}^{(elast.)} + \left.\frac{\delta f_{\mathfrak{e}}}{\delta t}\right|_{\mathfrak{e}\mathfrak{g}}^{(inelast.)} + \left.\frac{\delta f_{\mathfrak{e}}}{\delta t}\right|_{\mathfrak{e}\mathfrak{e}} + \left.\frac{\delta f_{\mathfrak{e}}}{\delta t}\right|_{\mathfrak{e}\mathfrak{i}}.$$

electron-gas elastic collisions

- Boltzmann operator
- Lorentz gas

$$\left.\frac{\delta f_{\mathfrak{e}}}{\delta t}\right|_{\mathfrak{e}\,\mathfrak{g}}^{(Boltz)} = \int\int \left(f_{\mathfrak{e}}'f_{\mathfrak{g}}' - f_{\mathfrak{e}}f_{\mathfrak{g}}\right) g\sigma d\Omega d\mathbf{v}_{\mathfrak{g}}.$$

$$\left.\frac{\delta f_{\mathfrak{e}}}{\delta t}\right|_{\mathfrak{e}\,\mathfrak{g}}^{(Lorentz)} = n_{\mathfrak{g}} v_{\mathfrak{e}} \int \left(f_{\mathfrak{e}}' - f_{\mathfrak{e}}\right) \sigma(v_{\mathfrak{e}}, \chi) d\Omega$$



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- Lorentz gas
- BGK operator

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$$\left.\frac{\delta f_{\mathfrak{e}}}{\delta t}\right|_{\mathfrak{e}\,\mathfrak{g}}^{(BGK)} = \nu_m (f_{\mathfrak{g}} - f_{\mathfrak{e}})$$

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electron-gas inelastic collisions

- Boltzmann operator
- Lorentz gas

$$\left.\frac{\delta f_{\mathfrak{e}}}{\delta t}\right|_{\mathfrak{e}\mathfrak{g}}^{(Boltz)} = \int\int \left(f'_{\mathfrak{e}} f'_{\mathfrak{g}} - f_{\mathfrak{e}} f_{\mathfrak{g}}\right) g \sigma d\Omega d\mathbf{v}_{\mathfrak{g}}.$$

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electron-gas inelastic collisions

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$$\left.\frac{\delta f_{\mathfrak{e}}}{\delta t}\right|_{\mathfrak{eg}}^{(Lorentz)} = n_{\mathfrak{g}} \int \left(\frac{v_{\mathfrak{e}}'^2}{v_{\mathfrak{e}}^2} f_{\mathfrak{e}}' \sigma(v_{\mathfrak{e}}', \Omega) - f_{\mathfrak{e}}\, \sigma(v_{\mathfrak{e}}, \Omega)\right) v_{\mathfrak{e}}\, d\Omega$$

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Coulomb collisions

- Landau operator
- Boltzmann operator (screened at Debye length)

$$\left.\frac{\delta f_{\mathfrak{e}}}{\delta t}\right|_{\mathfrak{eg}}^{(Fokker-Planck)} = \partial_{v_r}\left(D^{\alpha}_{rs}\partial_{v_s} f_{\mathfrak{e}}\right) - \partial_{v_r}\left(A^{\alpha}_{r} f_{\mathfrak{e}}\right) \quad \alpha \in \{\mathfrak{e}, \mathfrak{i}\}$$

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In this work we will use

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- Lorentz gas
- BGK operator

electron-gas inelastic collisions

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Under our conditions, the plasma is in a regime between the continuum and the kinetic descriptions

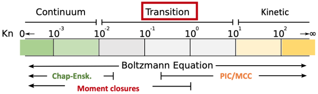

Figure: Adapted from Course by C. Groth

Electron collisional regimes

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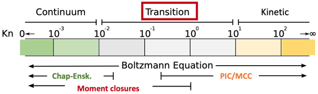

Figure: Adapted from Course by C. Groth

- The macroscopic time is defined as $t^0 = L^0/u^0$ where $u^0 = (eT_\mathfrak{e}^0/m_\mathfrak{h}^0)^{1/2}$ is the Bohm speed.
- The electron time is $t_\mathfrak{e}^0 = L^0/V_\mathfrak{e}^0$ where $V_\mathfrak{e}^0 = (eT_\mathfrak{e}^0/m_\mathfrak{e})^{1/2}$.

The scaled kinetic equation reads

$$\frac{\partial \tilde{f}_\mathfrak{e}}{\partial \tilde{t}} + \frac{1}{\varepsilon}\tilde{\boldsymbol{v}}\cdot\tilde{\nabla}\tilde{f}_\mathfrak{e} + \frac{1}{\varepsilon}\tilde{\nabla}\tilde{\phi}\cdot\tilde{\nabla}_{\tilde{\boldsymbol{v}}}\tilde{f}_\mathfrak{e} = \frac{1}{\varepsilon}\left(\frac{1}{\mathrm{Kn}_{\mathfrak{eg}}}\left.\frac{\delta \tilde{f}_\mathfrak{e}}{\delta \tilde{t}}\right|_{\mathfrak{eg}} + \frac{1}{\mathrm{Kn}_{\mathfrak{ee}}}\left.\frac{\delta \tilde{f}_\mathfrak{e}}{\delta \tilde{t}}\right|_{\mathfrak{ei}} + \frac{1}{\mathrm{Kn}_{\mathfrak{ee}}}\left.\frac{\delta \tilde{f}_\mathfrak{e}}{\delta \tilde{t}}\right|_{\mathfrak{ee}}\right).$$

where

$$\varepsilon = \sqrt{\frac{m_\mathfrak{e}}{m_\mathfrak{h}^0}},\quad \mathrm{Kn}_{\mathfrak{eg}} = \frac{1}{n_\mathfrak{g}^0\sigma_{\mathfrak{eg}}^0 L} \quad \text{and} \quad \mathrm{Kn}_{\mathfrak{ee}} = \frac{1}{n_\mathfrak{e}^0\sigma_{\mathfrak{ee}}^0 L}$$

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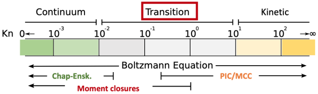

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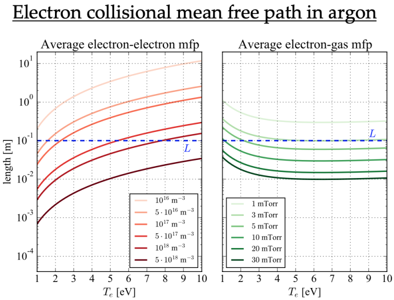

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where

$$\varepsilon = \sqrt{\frac{m_{\mathfrak{e}}}{m^0_{\mathfrak{h}}}}, \quad \mathrm{Kn}_{\mathfrak{eg}} = \frac{1}{n^0_{\mathfrak{g}}\sigma^0_{\mathfrak{eg}}L} \quad \text{and} \quad \mathrm{Kn}_{\mathfrak{ee}} = \frac{1}{n^0_{\mathfrak{e}}\sigma^0_{\mathfrak{ee}}L}$$

We are particularly interested in properly representing the electron energy distribution function (EEDF). This is defined as:

$$g_{\mathfrak{e}}(\mathcal{E},\boldsymbol{x},t)d\mathcal{E} = v_{\mathfrak{e}}^2 dv \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \, f_e(\boldsymbol{v}_{\mathfrak{e}},\boldsymbol{x},t) \, d\varphi \tag{2}$$

where the velocity in polar coordinates reads $\boldsymbol{v}_{\mathfrak{e}} = v_{\mathfrak{e}} \left(\cos\varphi, \sin\varphi\cos\theta, \sin\varphi\sin\theta\right)$ and the energy is $e\mathcal{E} = \frac{1}{2} m_{\mathfrak{e}} v_{\mathfrak{e}}^2$.

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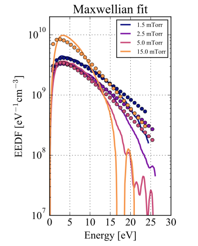
Maxwellian



$$f_{\mathfrak{e}}^{(M)}(\mathbf{v}, \mathbf{x}, t) = n_{\mathfrak{e}} \left(\frac{\beta_{\mathfrak{e}}}{\pi}\right)^{3/2} \exp\left(-\beta_{\mathfrak{e}} v^2\right)$$

- Not able to represent deviations in the tail of the EEDF

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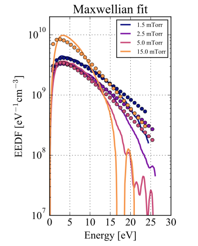
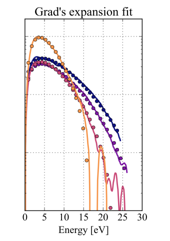


$$f_e^{(M)}(\mathbf{v}, \mathbf{x}, t) = n_e \left(\frac{\beta_e}{\pi}\right)^{3/2} \exp\left(-\beta_e v^2\right)$$

- Not able to represent deviations in the tail of the EEDF

Grad's



$$f^{(Grad)}(c_i) = f^{(M)}(c_i)\,(1 + A + A_i c_i + B_{ij} c_i c_j + D_{ijk} c_i c_j c_k + \ldots)$$

- Problems: Hyperbolicity and Positivity.

Non-equilibrium distribution functions

We are particularly interested in properly representing the electron energy distribution function (EEDF). This is defined as:

$$g_{\mathfrak{e}}(\mathcal{E}, \mathbf{x}, t) d\mathcal{E} = v_{\mathfrak{e}}^2 dv \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \, f_e(\mathbf{v}_{\mathfrak{e}}, \mathbf{x}, t) \, d\varphi \tag{2}$$

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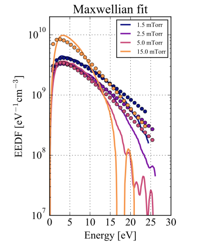
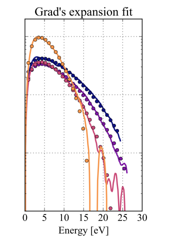
Maxwellian



$$f_{\mathfrak{e}}^{(M)}(\mathbf{v}, \mathbf{x}, t) = n_{\mathfrak{e}} \left(\frac{\beta_{\mathfrak{e}}}{\pi} \right)^{3/2} \exp\left(-\beta_{\mathfrak{e}} v^2 \right)$$

- Not able to represent deviations in the tail of the EEDF

Grad's



$$f^{(Grad)}(c_i) = f^{(M)}(c_i) \left(1 + A + A_i c_i + B_{ij} c_i c_j + D_{ijk} c_i c_j c_k + \ldots \right)$$

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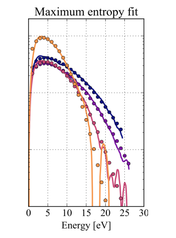
Maximum entropy



$$f^{(Max.entr.)}(c_i) = \exp\left(A + A_i c_i + B_{ij} c_i c_j + D_{ijk} c_i c_j c_k + \ldots \right)$$

- Problems: Generalization to different operators.

Grad's closure with skewness and excess kurtosis perturbations

Moment equations

In order to have skewness and excess kurtosis perturbations we choose the following moment system:

$$\frac{\partial}{\partial t}\int_\infty \psi f_e d\mathbf{v}_e + \nabla\cdot\int_\infty \psi \mathbf{v} f_e d\mathbf{v}_e - \int_\infty f_e \frac{e\nabla\phi}{m_e}\cdot\nabla_{\mathbf{v}}\psi d\mathbf{v}_e = \int_\infty \psi \left.\frac{\delta f_e}{\delta t}\right|_c d\mathbf{v}_e .$$

with the weights

$$\psi(\mathbf{v}_e) = \left(m_e,\ m_e\mathbf{v}_e,\ \frac{m_e}{2}c_e^2,\ \frac{m_e}{2}c_e^2\mathbf{c}_e,\ \frac{m_e}{2}c_e^4\right)^T .$$

where $\mathbf{c}_e = \mathbf{v}_e - \mathbf{u}_e$ is the the peculiar velocity.

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where $\mathbf{c}_e = \mathbf{v}_e - \mathbf{u}_e$ is the the peculiar velocity.
These macroscopic variables are defined as follows,

$$\textbf{Density}\ \ n_e = \int_{\infty} f_e d\mathbf{v},\quad \textbf{Flux}\ \ \rho_e u_{e_i} = \int_{\infty} m_e v_i f_e d\mathbf{v},\quad \textbf{Internal energy}\ \ p_e = \frac{1}{3}\int_{\infty} m_e c_e^2 f_e d\mathbf{v}, \tag{3}$$

$$\textbf{Heat flux}\ \ q_{e_i} = \frac{1}{2}\int_{\infty} m_e c_e^2 c_{e_i} f_e d\mathbf{v},\quad \text{and}\quad \textbf{Fourth moment}\ \ p_{e_{iijj}} = \frac{1}{2}\int_{\infty} m_e c_e^4 f_e d\mathbf{v}.$$

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These macroscopic variables are defined as follows,

$$\textbf{Density}\ \ n_e = \int_\infty f_e d\mathbf{v},\quad \textbf{Flux}\ \ \rho_e u_{e_i} = \int_\infty m_e v_i f_e d\mathbf{v},\quad \textbf{Internal energy}\ \ p_e = \frac{1}{3}\int_\infty m_e c_e^2 f_e d\mathbf{v}, \tag{3}$$

$$\textbf{Heat flux}\ \ q_{e_i} = \frac{1}{2}\int_\infty m_e c_e^2 c_{e_i} f_e d\mathbf{v},\quad \text{and}\quad \textbf{Fourth moment}\ \ p_{e_{iijj}} = \frac{1}{2}\int_\infty m_e c_e^4 f_e d\mathbf{v}.$$

The system of equations reads

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x_i} n_e u_{e_i} = \dot{n}_e, \tag{4}$$

$$m_e \frac{\partial}{\partial t} n_e u_{e_i} + \frac{\partial}{\partial x_j}\left(m_e n_e u_{e_i} u_{e_j} + p_e\delta_{ij}\right) = -e n_e E_i + R_i, \tag{5}$$

$$\frac{3}{2}\frac{\partial p_e}{\partial t} + \frac{\partial}{\partial x_k}\left(q_{e_k} + \frac{3}{2}p_e u_{e_k}\right) + p_e\frac{\partial u_{e_k}}{\partial x_k} = Q, \tag{6}$$

$$\frac{\partial q_{e_i}}{\partial t} + \frac{\partial}{\partial x_j}\left(r_{e_{ij}} + q_{e_i}u_{e_j}\right) + r_{e_{ijk}}\frac{\partial u_{e_k}}{\partial x_j} + q_{e_j}\frac{\partial u_{e_i}}{\partial x_j} - \frac{5}{2}\frac{p_e}{\rho_e}\frac{\partial p_e}{\partial x_j}\delta_{ij} = R_i^{hf} - \frac{5}{2}\frac{p_e}{\rho_e}\left(R_i - m_e\dot{n}_e u_{e_i}\right), \tag{7}$$

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In order to have skewness and excess kurtosis perturbations we choose the following moment system:

$$\frac{\partial}{\partial t}\int_\infty \psi f_e d\mathbf{v}_e + \nabla\cdot\int_\infty \psi \mathbf{v} f_e d\mathbf{v}_e - \int_\infty f_e \frac{e\nabla\phi}{m_e}\cdot\nabla_{\mathbf{v}}\psi d\mathbf{v}_e = \int_\infty \psi \left.\frac{\delta f_e}{\delta t}\right|_c d\mathbf{v}_e .$$

with the weights

$$\psi(\mathbf{v}_e) = \left(m_e,\ m_e\mathbf{v}_e,\ \frac{m_e}{2}c_e^2,\ \frac{m_e}{2}c_e^2\mathbf{c}_e,\ \frac{m_e}{2}c_e^4\right)^T .$$

where $\mathbf{c}_e = \mathbf{v}_e - \mathbf{u}_e$ is the the peculiar velocity.
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By injecting the Grad's ansantz in the definition of the variables we obtain the following distribution function

$$f_{\mathfrak{e}}(\boldsymbol{x}, \boldsymbol{c}_{\mathfrak{e}}, t) = f_{\mathfrak{e}}^{(M)} \left\{ 1 + \frac{8\beta_{\mathfrak{e}}^2}{5\rho_{\mathfrak{e}}} q_{\mathfrak{e}_i} c_{\mathfrak{e}_i} \left(\beta_{\mathfrak{e}} c_{\mathfrak{e}}^2 - \frac{5}{2} \right) + \left(\frac{15}{8} - \frac{5\beta_{\mathfrak{e}}}{2} c_{\mathfrak{e}}^2 + \frac{\beta_{\mathfrak{e}}^2}{2} c_{\mathfrak{e}}^4 \right) \Delta_{\mathfrak{e}} \right\}.$$

where the non-dimensional excess kurtosis is defined as

$$\Delta_{\mathfrak{e}} = \frac{p_{\mathfrak{e}_{iijj}} - p_{\mathfrak{e}_{iijj}}^{(M)}}{p_{\mathfrak{e}_{iijj}}^{(M)}} = \frac{2}{15} \frac{\rho_{\mathfrak{e}}}{p_{\mathfrak{e}}^2} \int_{\infty} m_{\mathfrak{e}} c_{\mathfrak{e}}^4 \left(f_{\mathfrak{e}} - f_{\mathfrak{e}}^{(M)} \right) d\boldsymbol{v}$$

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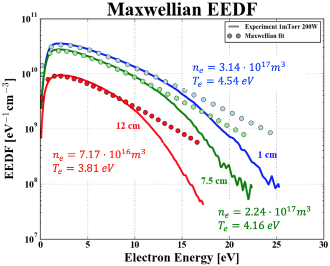
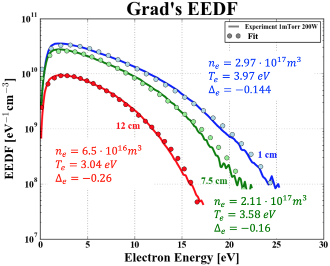
The new variable is able to parametrize the electron energy distribution function





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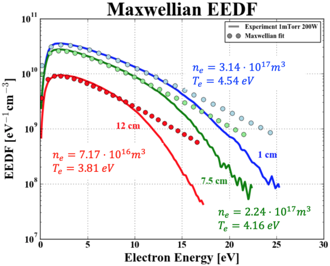

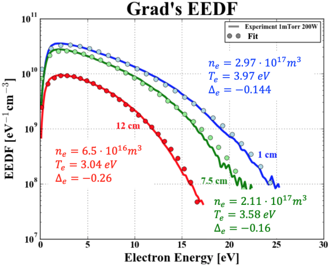

Note: Grad's 13M model has a Maxwellian EEDF

We use the following properties and definitions

- Conservation: $m_{\mathfrak{e}}\boldsymbol{v}_{\mathfrak{e}} + m_{\mathfrak{g}}\boldsymbol{v}_{\mathfrak{g}} = m_{\mathfrak{e}}\boldsymbol{v}'_{\mathfrak{e}} + m_{\mathfrak{g}}\boldsymbol{v}'_{\mathfrak{g}}$ and $\frac{1}{2}m_{\mathfrak{e}}v_{\mathfrak{e}}^2 + \frac{1}{2}m_{\mathfrak{g}}v_{\mathfrak{g}}^2 = \frac{1}{2}m_{\mathfrak{e}}v_{\mathfrak{e}}'^2 + \frac{1}{2}m_{\mathfrak{g}}v_{\mathfrak{g}}'^2$
- Mom. coll. operator: $\int_{\infty} \psi_{\mathfrak{e}} \left.\frac{\delta f_{\mathfrak{e}}}{\delta t}\right|_{c} d\boldsymbol{v} = \int_{\infty}\int_{\infty}\int (\psi'_{\mathfrak{e}} - \psi_{\mathfrak{e}}) f_{\mathfrak{e}} f_{\mathfrak{g}}\, g\sigma\, d\Omega\, d\boldsymbol{v}_{\mathfrak{g}}\, d\boldsymbol{v}_{\mathfrak{e}}\,.$
- Centre-of-mass vars: $\boldsymbol{G} := \frac{m_{\mathfrak{e}}\boldsymbol{v}_{\mathfrak{e}} + m_{\mathfrak{g}}\boldsymbol{v}_{\mathfrak{g}}}{m_{\mathfrak{e}} + m_{\mathfrak{g}}}, \quad \boldsymbol{g} := \boldsymbol{v}_{\mathfrak{e}} - \boldsymbol{v}_{\mathfrak{g}}\,.$

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We write the collisional integrals in the centre-of-mass variables

- Momentum exchange

$$\mathbf{R}_{\mathfrak{e}\,\mathfrak{g}}^{(el)} = -\mu_{\mathfrak{e}\,\mathfrak{g}} \int_\infty \int_\infty \mathbf{g} Q^{(1)} f_{\mathfrak{e}} f_{\mathfrak{g}}\, g d\mathbf{g} d\mathbf{G}. \tag{9}$$

- Energy exchange

$$Q_{\mathfrak{e}\,\mathfrak{g}_{Tot}}^{(el)} = -\mu_{\mathfrak{e}\,\mathfrak{g}} \int_\infty \int_\infty (\mathbf{G}\cdot\mathbf{g})\, Q^{(1)} f_{\mathfrak{e}} f_{\mathfrak{g}}\, g d\mathbf{g} d\mathbf{G}. \tag{10}$$

- Heat flux exchange

$$\begin{aligned}\mathbf{R}_{\mathfrak{e}\,\mathfrak{g}_{Tot}}^{hF,(el)} = -\frac{\mu_{\mathfrak{e}\,\mathfrak{g}}}{2} \int_\infty \int_\infty \Bigg\{ &\left[(\mathbf{G}\cdot\mathbf{g})\mathbf{G} + G^2\mathbf{g} + \left(\frac{\mu_{\mathfrak{e}\,\mathfrak{g}}}{m_{\mathfrak{e}}}\right)^2 g^2 \mathbf{g} \right] Q^{(1)} \\ &-2\left(\frac{\mu_{\mathfrak{e}\,\mathfrak{g}}}{m_{\mathfrak{e}}}\right) \left[\frac{1}{2} g^2 \mathbf{G} - \frac{3}{2}(\mathbf{G}\cdot\mathbf{g})\mathbf{g} \right] Q^{(2)} \Bigg\} g f_{\mathfrak{e}} f_{\mathfrak{g}}\, d\mathbf{g} d\mathbf{G}.\end{aligned} \tag{11}$$

- Fourth moment exchange

$$\begin{aligned}Q_{\mathfrak{e}\,\mathfrak{g}_{Tot}}^{(4)} = -2\mu_{\mathfrak{e}\,\mathfrak{g}} \int_\infty \int_\infty \Bigg\{ &\left[G^2 + \left(\frac{\mu_{\mathfrak{e}\,\mathfrak{g}}}{m_{\mathfrak{e}}}\right)^2 g^2 \right] (\mathbf{G}\cdot\mathbf{g})\, Q^{(1)} \\ &+ \left(\frac{\mu_{\mathfrak{e}\,\mathfrak{g}}}{m_{\mathfrak{e}}}\right) \left[\frac{3}{2}(\mathbf{G}\cdot\mathbf{g})^2 - \frac{1}{2} g^2 G^2 \right] Q^{(2)} \Bigg\} g f_{\mathfrak{e}} f_{\mathfrak{g}}\, d\mathbf{g} d\mathbf{G}.\end{aligned} \tag{12}$$

We obtain the following results

- Momentum exchange

$$\boldsymbol{R}_{\mathfrak{e}\,\mathfrak{g}}^{(el)} = -m_{\mathfrak{e}} n_{\mathfrak{e}} \nu_{\mathfrak{e}\,\mathfrak{g}}^{(fr,1)} \boldsymbol{u}_{\mathfrak{e}} - \underbrace{m_{\mathfrak{e}} n_{\mathfrak{e}} \nu_{\mathfrak{e}\,\mathfrak{g}}^{(skew,1)} \frac{\boldsymbol{q}_{\mathfrak{e}}}{p_{\mathfrak{e}}}}_{\text{Soret effect}} .$$

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$$Q_{\mathfrak{e}\,\mathfrak{g}}^{(el)} = \frac{m_{\mathfrak{e}}}{m_{\mathfrak{g}}} n_{\mathfrak{e}} \nu_{\mathfrak{e}\,\mathfrak{g}}^{(fr,2)} e\,(T_{\mathfrak{g}} - T_{\mathfrak{e}}) - \underbrace{\frac{m_{\mathfrak{e}}}{m_{\mathfrak{g}}} n_{\mathfrak{e}} \nu_{\mathfrak{e}\,\mathfrak{g}}^{(kurt,2)} \Delta_{\mathfrak{e}} e T_{\mathfrak{g}}}_{\text{Kurtosis correction}} - \boldsymbol{R}_{\mathfrak{e}\,\mathfrak{g}}^{(el)} \cdot \boldsymbol{u}_{\mathfrak{e}} .$$

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$$\boldsymbol{R}_{\mathfrak{e}\,\mathfrak{g}}^{hF,(el)} = - \underbrace{n_{\mathfrak{e}} \nu_{\mathfrak{e}\,\mathfrak{g}}^{(fr,3)} e T_{\mathfrak{e}} \boldsymbol{u}_{\mathfrak{e}}}_{\text{Dufour effect}} - \nu_{\mathfrak{e}\,\mathfrak{g}}^{(skew,3)} \boldsymbol{q}_{\mathfrak{e}} .$$

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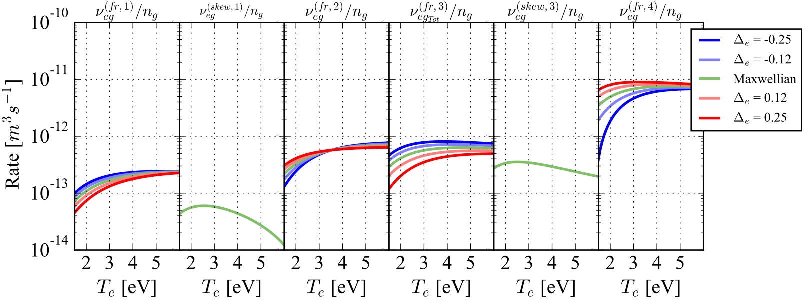
- Fourth moment exchange

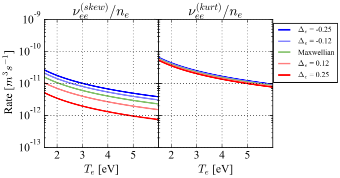
$$Q_{\mathfrak{e}\,\mathfrak{g}}^{(el,4)} = \frac{m_{\mathfrak{e}}}{m_{\mathfrak{g}}} \nu_{\mathfrak{e}\,\mathfrak{g}}^{(fr,4)} \frac{p_{\mathfrak{e}}^2}{\rho_{\mathfrak{e}}} \left(\frac{T_{\mathfrak{g}}}{T_{\mathfrak{e}}} - 1\right) - \underbrace{\frac{m_{\mathfrak{e}}}{m_{\mathfrak{g}}} \nu_{\mathfrak{e}\,\mathfrak{g}}^{(kurt,4)} \Delta_{\mathfrak{e}} \frac{p_{\mathfrak{e}}^2}{\rho_{\mathfrak{e}}} \frac{T_{\mathfrak{g}}}{T_{\mathfrak{e}}}}_{\text{Kurtosis correction}} + 4\nu_{\mathfrak{e}\,\mathfrak{g}}^{(skew,3)} \boldsymbol{q}_{\mathfrak{e}} \cdot \boldsymbol{u}_{\mathfrak{e}} .$$



The frequencies are functions of the Chapman-Cowling integrals

$$\Omega_{\mathfrak{eg}}^{(l,r)}(T_{\mathfrak{e}}) = \frac{1}{2}\left(\frac{1}{\pi\beta_{\mathfrak{e}}}\right)^{1/2}\int_0^\infty \xi^{2r+3}e^{-\xi^2}Q^{(l)}d\xi \quad \text{with} \quad \xi = \sqrt{\beta_{\mathfrak{e}}}g. \tag{13}$$

Closure of collisional integrals: electron-electron and ionization collisions



electron-electron collisions:

- Heat flux exchange

$$\boldsymbol{R}_{ee}^{hF} = -\nu_{ee}^{(skew)}\boldsymbol{q}_e \,.$$

- Fourth-moment exchange

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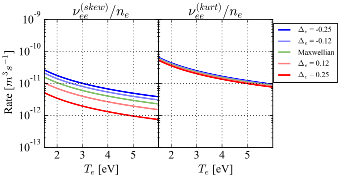

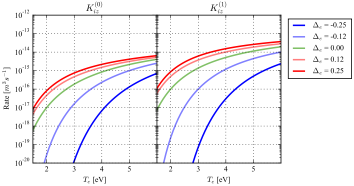

electron-electron collisions:

- Heat flux exchange

$$\boldsymbol{R}_{\mathfrak{e}\mathfrak{e}}^{hF} = -\nu_{\mathfrak{e}\mathfrak{e}}^{(skew)} \boldsymbol{q}_{\mathfrak{e}} .$$

- Fourth-moment exchange

$$Q_{\mathfrak{e}\mathfrak{e}}^{(4)} = -\nu_{\mathfrak{e}\mathfrak{e}}^{(kurt)} \frac{p_{\mathfrak{e}}^2}{\rho_{\mathfrak{e}}} \Delta_{\mathfrak{e}} .$$

Inelastic collisions

- Density production

$$\dot{n}_{\mathfrak{e}}^{(iz)} = n_{\mathfrak{e}} n_{\mathfrak{g}} K_{iz}^{(0)} .$$

- Energy losses

$$Q_{\mathfrak{e}\mathfrak{g}}^{(inel)} = -\sum_{k=0}^{excit,iz} n_{\mathfrak{e}} n_{\mathfrak{g}} K_{inel,k}^{(0)} \phi_k^* ,$$

- Fourth-moment losses

$$Q_{\mathfrak{e}\mathfrak{g}}^{(inel,4)} = -2 \left(\frac{p_{\mathfrak{e}}^2}{\rho_{\mathfrak{e}}} \right) \sum_{k=0}^{excit,iz} \left(K_{inel,k}^{(0)} \left(\frac{\phi_k^*}{T_e} \right)^2 + K_{inel,k}^{(1)} \left(\frac{\phi_k^*}{T_e} \right) \right) .$$

Numerical results

Case 1: Relaxation in a homogeneous background

We study a 0D argon plasma where the electrons are initially at 5 eV and Maxwellian distribution and the gas at room temperature (0.026 eV)

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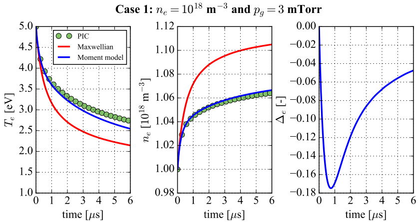

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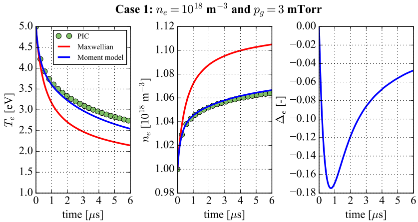
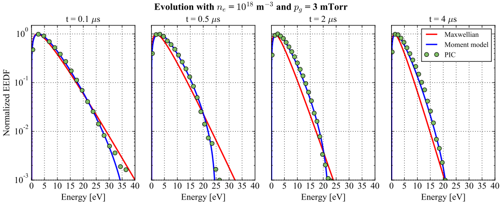
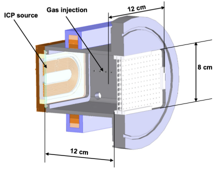


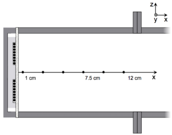


CASE 2: 1D SIMULATION OF AN ICP REACTOR

We study a 1D slab along the axis of the ICP reactor working on argon.

- We solve a model with a finite volume scheme with:
  - Lax-Friedrichs scheme
  - TVD reconstruction with ospre limiter
  - Third-order TVD Runge-Kutta in time





We study a 1D slab along the axis of the ICP reactor working on argon.

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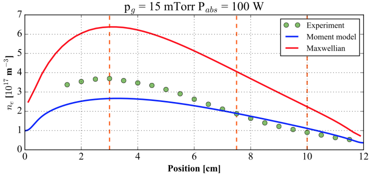

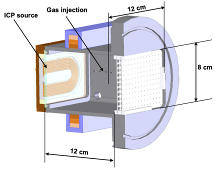

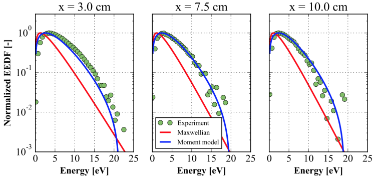

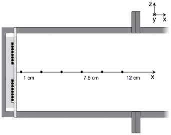

Future work and conclusions

Open questions and future work

There is a number of open questions that can provide interdisciplinary interaction between mathematics and physics:

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  - Regularization of the equations
  - Extension of Maximum-Entropy to multi-component and general collisional operator?

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If we consider the BGK operator

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However, with our approach, we have additional effects

$$\begin{aligned}
\boldsymbol{u}_{\mathfrak{e}}^{C-E} &= -D_{\mathfrak{e}}\left(\frac{1}{n_{\mathfrak{e}}}\nabla n_{\mathfrak{e}} + (1 + \underbrace{\chi_{\mathfrak{e}}}_{\text{thermodiff.}})\nabla \ln T_{\mathfrak{e}} + \underbrace{\alpha_{\mathfrak{e}}\nabla\Delta_{\mathfrak{e}}}_{\text{Diffusion due to EEDF}}\right) - \mu_{\mathfrak{e}}\boldsymbol{E}, \\
\boldsymbol{h}_{\mathfrak{e}}^{C-E} &= \beta_{\mathfrak{e}} n_{\mathfrak{e}} \boldsymbol{u}_{\mathfrak{e}}^{C-E} - \kappa_{\mathfrak{e}}\nabla T_{\mathfrak{e}} - \underbrace{\vartheta_{\mathfrak{e}}\nabla n_{\mathfrak{e}}}_{\text{Dufour}} - \underbrace{\varkappa_{\mathfrak{e}}\nabla\Delta_{\mathfrak{e}}}_{\text{Diffusion due to EEDF}} .
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- Paper with derivation of the model and comparison to experiments in preparation.

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- **UTIAS**: C. Groth
- **VKI**: T. Magin, S. Bocelli

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