The stochastic Zakharov system in dimension 1

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We work on the stochastic Zakharov system with a damping term :

$$(SZ^{\varepsilon})\begin{cases} i\partial_t u = -\partial_x^2 u + nu \\ \varepsilon^2 d(\partial_t n) + \alpha\varepsilon\partial_t n = \partial_x^2(n + |u|^2)dt + \phi dW_t \end{cases} \tag{1}$$

with initial data u_0, n_0, n_1. W_t is a cylindrical Wiener process : let $(e_k)_{k\in\mathbb{N}}$ be an Hilbertian basis, then we define W as

$$W(\omega, t, x) = \sum_{k\in\mathbb{N}} e_k(x)\beta_k(\omega, t).$$

For the Zakharov system, mass and energy are preserved :

$$N = \|u\|_{L^2}^2, \qquad H = \|\partial_x u\|_{L^2}^2 + \frac{1}{2}\left(\|n\|_{L^2}^2 + \left\|\varepsilon\partial_x^{-1}\partial_t n\right\|_{L^2}^2\right) + \int_{\mathbb{R}} n|u|^2 dx.$$

Theorem

Let $\alpha \geqslant 0$. For initial data $u_0 \in H^3$, $n_0 \in H^2 \cap \dot{H}^{-1}$ and $n_1 \in H^1 \cap \dot{H}^{-1}$, there exists almost surely a solution $(u, n) \in L^\infty(\mathbb{R}^+, H^3 \times H^2)$ for the system (SZ^ε).

idea of the proof

We write $n = m + Z^\varepsilon$ where Z^ε satisfies :

$$\varepsilon^2 d(\partial_t Z^\varepsilon) + \alpha\varepsilon\partial_t Z^\varepsilon = \partial_x^2 Z^\varepsilon dt + \phi dW_t \tag{2}$$

(u, n) is solution of (SZ^ε) if and only if (u, m) is solution of

$$\begin{cases} i\partial_t u = \partial_x^2 u + (m + Z^\varepsilon)u \\ \\ \varepsilon^2\partial_t^2 m + \alpha\varepsilon\partial_t m = \partial_x^2(m + |u|^2) \end{cases} \tag{3}$$

Limit $\varepsilon \to 0$

We need to rescale the system (good scaling : $Z^\varepsilon = \frac{1}{\sqrt{\varepsilon}} z(\frac{t}{\varepsilon})$) :

$$(S\tilde{Z}^\varepsilon) \begin{cases} i\partial_t u = \partial_x^2 u + mu + \frac{1}{\sqrt{\varepsilon}} z(\frac{t}{\varepsilon}) u \\ \\ \varepsilon^2 \partial_t^2 m + \alpha\varepsilon\partial_t m = \partial_x^2 (m + |u|^2) \end{cases} \quad (4)$$

where z satisfies $d(\partial_t z_t) + \alpha\partial_t z = \partial_x^2 z_t dt + \phi dW_t$.

Benefits:

- The process z does not depend on ε, and is stationnary.
- we control the growth of $z(\frac{t}{\varepsilon})$: for every $\delta > 0$ there exists a stopping time τ_δ^ε such that for $t < \tau_\delta^\varepsilon$: $\left\| z(\frac{t}{\varepsilon}) \right\|_E \leqslant \varepsilon^{-\delta}$.

Here there is not conservation of the energy :

$$\partial_t H(u, m) = -\int_{\mathbb{R}} Z^\varepsilon \partial_t |u|^2 dx - \alpha\varepsilon \left\| \partial_x^{-1} \partial_t m \right\|_{L^2}. \quad (5)$$

$$(SZ^{\varepsilon})\begin{cases} i\partial_t u = -\partial_x^2 u + nu \\ \varepsilon^2 d(\partial_t n) + \alpha\varepsilon\partial_t n = \partial_x^2(n+|u|^2)dt + \phi dW_t \end{cases}$$

$$(S\tilde{Z}^{\varepsilon})\begin{cases} i\partial_t u = \partial_x^2 u + mu + \frac{1}{\sqrt{\varepsilon}} z(\frac{t}{\varepsilon})u \\ \varepsilon^2 \partial_t^2 m + \alpha\varepsilon\partial_t m = \partial_x^2(m+|u|^2) \end{cases}$$

where z is solution of $d(\partial_t z_t) + \alpha\partial_t z = \partial_x^2 z_t dt + \phi dW_t$.

Theorem

For any $T > 0$, and for $(u_0, m_0, m_1) \in H^3 \times (H^2 \cap \dot{H}^{-1}) \times (H^1 \cap \dot{H}^{-1})$, the process u^{ε} solution of the system $(S\tilde{Z}^{\varepsilon})$ with $(z, \zeta) \in (H^3 \cap \dot{H}^{-3})^2$ converges in law in $C([0,T], H^s_{loc})$ for $s < 1$ to u solution of

$$idu = (-\partial_x^2 u - |u|^2 u - \frac{i}{2} uF)dt - u(\partial_x^2)^{-1}\phi dW_t, \quad (6)$$

where $F(x) = \sum_{k\in\mathbb{N}} \left((\partial_x^2)^{-1}\phi e_k(x)\right)^2$.

Transition semigroup and infinitesimal generator

Definition

Let X_t be a continuous random variable.We call transition semigroup of X_t the operator P_t such that for all continuous function f :

$$P_t f(x) = \mathbb{E}_{X_0 = x}[f(X_t)].$$

We call infinitesimal generator of X_t the operator defined by

$$\mathscr{L} f(x) = \lim_{t \to 0} \frac{P_t f(x) - f(x)}{t}.$$

Example

For a SDE : $dX_t = a(X_t)dt + \sigma(X_t)dB_t$ the infinitesimal generator is

$$\mathscr{L}\varphi(x) = a(x)\varphi'(x) + \frac{1}{2}\sigma^2(x)\varphi''(x).$$

Perturbed Test Function method

Let x^ε solution of :

$$\partial_t x^\varepsilon + \lambda x^\varepsilon = \frac{1}{\sqrt{\varepsilon}} x^\varepsilon z\left(\frac{t}{\varepsilon}\right). \tag{7}$$

We define $z^\varepsilon(t) = z(\frac{t}{\varepsilon})$. We denote by $\mathscr{M}$ the infinitesimal generator of z, and $\mathscr{L}^\varepsilon$ the infinitesimal generator of $(x^\varepsilon, z^\varepsilon)$. Let $\varphi(x) = \frac{1}{2}|x|^2$. Then

$$\mathscr{L}^\varepsilon \varphi(x^\varepsilon) = \frac{1}{\varepsilon}\mathscr{M}\varphi(x^\varepsilon) - \lambda |x^\varepsilon|^2 + \frac{1}{\sqrt{\varepsilon}}|x^\varepsilon|^2 z\left(\frac{t}{\varepsilon}\right).$$

We add a corrector $\sqrt{\varepsilon}\varphi_1$:

$$\begin{aligned}\mathscr{L}^\varepsilon(\varphi + \sqrt{\varepsilon}\varphi_1)(x^\varepsilon, z^\varepsilon) = &- \lambda |x^\varepsilon|^2 + \frac{1}{\sqrt{\varepsilon}}\left(|x^\varepsilon|^2 z\left(\frac{t}{\varepsilon}\right) + \mathscr{M}\varphi_1(x^\varepsilon, z^\varepsilon)\right) \\ &+ D_x\varphi_1(x^\varepsilon z^\varepsilon) + \sqrt{\varepsilon} D_x \varphi_1(-\lambda |x^\varepsilon|^2).\end{aligned}$$

Thus we choose : $\varphi_1(x,v) = -|x|^2 \mathscr{M}^{-1} v$. We get

$$\mathscr{L}^\varepsilon(\varphi + \sqrt{\varepsilon}\varphi_1)(x^\varepsilon, z^\varepsilon) = -\lambda|x^\varepsilon|^2 - 2|x^\varepsilon|^2 z^\varepsilon \mathscr{M}^{-1} z^\varepsilon + \lambda\sqrt{\varepsilon}|x^\varepsilon|^2 \mathscr{M}^{-1} z^\varepsilon.$$

We need a second corrector $\varepsilon\varphi_2$, satisfying

$$\mathscr{M}\varphi_2(x^\varepsilon, z^\varepsilon) = 2|x^\varepsilon|^2 \left(z^\varepsilon \mathscr{M}^{-1} z^\varepsilon - \mathbb{E}_\nu \left[z\mathscr{M}^{-1} z \right] \right).$$

Defining $\varphi^\varepsilon(x,v) = \varphi(x) + \sqrt{\varepsilon}\varphi_1(x,v) + \varepsilon\varphi_2(x,v)$ we get

$$\begin{aligned} \mathscr{L}^\varepsilon \varphi^\varepsilon(x^\varepsilon, z^\varepsilon) = & -\lambda|x^\varepsilon|^2 - 2|x^\varepsilon|^2 \mathbb{E}_\nu \left[z\mathscr{M}^{-1} z \right] \\ & + \sqrt{\varepsilon} \left(\lambda|x^\varepsilon|^2 \mathscr{M}^{-1} z^\varepsilon + D_x\varphi_2(x^\varepsilon z^\varepsilon) \right) \\ & + \varepsilon D_x \varphi_2(-\lambda|x^\varepsilon|^2). \end{aligned}$$

Energy estimate

Energy : $H = \|\partial_x u\|_{L^2}^2 + \frac{1}{2}\left(\|m\|_{L^2}^2 + \left\|\varepsilon\partial_x^{-1}\partial_t m\right\|_{L^2}^2\right) + \int_{\mathbb{R}} m|u|^2 dx$,

We define $K = \frac{1}{2}\|\partial_x u\|_{L^2}^2 + \frac{1}{4}\|m\|_{L^2}^2 + \frac{1}{2}\left\|\varepsilon\partial_x^{-1}\partial_t m\right\|_{L^2}^2$.

Proposition

Let the initial conditions $(u_0, m_0, m_1, z_0, \zeta_0)$ be in $H^3 \times H^2 \times (H^1 \cap \dot{H}^{-1}) \times (H^3 \cap \dot{H}^{-2})^2$, and $\phi \in \mathscr{L}_2(H^3 \cap \dot{H}^{-3})$.Then there exists a constant $C(T) > 0$ which is independent of ε and a stopping time τ^ε such that the solution (u, m) of the system (4) satisfies

$$\mathbb{E}[\sup_{t \leqslant \tau_\varepsilon}(K(t))^2] \leqslant C(T).$$

- apply the Perturbed Test Function method to H,
- get a bound on $\mathscr{L}^\varepsilon(H + \sqrt{\varepsilon}H_1 + \varepsilon H_2)$,
- conclude with martingale inequalities.

Convergence in the martingale problem

We define the function $\varphi : u \mapsto (u,h)^\ell$ for $h \in H^1$ a fixed function with compact support, and $\ell = 1, 2$. As in the previous sections, we compute the generator $\mathscr{L}^\varepsilon \varphi$ and then correct it with two correctors.We get that

$$\varphi^\varepsilon(u^\varepsilon(t)) - \varphi^\varepsilon(u_0) - \int_0^t \mathscr{L}^\varepsilon \varphi^\varepsilon(u^\varepsilon(s))ds$$

is a martingale. Then we pass to the limit to show that

$$\varphi(u_t) - \varphi(u_0) - \int_0^t \mathscr{L}\varphi(u_s)ds$$

is a martingale. Finally, we identify the limit generator $\mathscr{L}\varphi$ and "read" the equaton satisfied by u.

The limit process u is a martingale solution of

$$\varphi(\tilde{u}_t) - \varphi(u_0) - \int_0^t D_u\varphi \left(i\partial_x^2 \tilde{u}_s + i|\tilde{u}_s|^2 \tilde{u}_s - \frac{1}{2}\tilde{u}_s F \right) ds = \int_0^t i\tilde{u}_s (\partial_x^2)^{-1} \phi dW_s.$$

$$\begin{array}{ccccc} \mathscr{L}^\varepsilon \varphi & \longrightarrow & \mathscr{L}^\varepsilon \varphi^\varepsilon & \longrightarrow & \mathscr{L}\varphi \\ \uparrow & & & & \downarrow \end{array}$$

Stochastic Zakharov system $\longrightarrow$ Stochastic Nonlinear Schrödinger equation

Thank you for your attention.

Improvement of the convergence

Proposition

The process u^ε converges in probability to the process u solution of

$$idu = (-\partial_x^2 u - |u|^2 u - \frac{i}{2} uF)dt - u(\partial_x^2)^{-1}\phi dW_t, \quad (8)$$

where $F(x) = \sum_{k\in\mathbb{N}} \left((\partial_x^2)^{-1}\phi e_k(x)\right)$ and W_t is the Wiener process introduced in (SZ^ε).

We know that for $\varphi(u) = (u, h)$

$$\varphi^\varepsilon(u_t^\varepsilon) - \varphi^\varepsilon(u_0) - \int_0^t \mathscr{L}^\varepsilon \varphi^\varepsilon(u_s^\varepsilon) ds = \int_0^t \left(iu^\varepsilon(\partial_x^2)^{-1}\phi dW_s, h\right) + G(\varepsilon) \quad (9)$$

with $G(\varepsilon) \to 0$.