Metastability for a system of interacting neurons

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Work in collaboration with Eva Löcherbach

The model

- N neurons, described by their membrane potential $u \geqslant 0$.
- At rate $\lambda(u) \geqslant 0$, a neuron emits a *spike* :

$$\mathbb{P}(\text{emits a spike during } [0,t]) \ = \ \lambda(u)\,t + \underset{t\to 0}{o}(t).$$

  Its potential is reseted to its resting value ($u = 0$) and all post-synaptic neurons get a fixed increase of their potential.
- Mean-field framework : all neurons are connected. When a neuron emits a spike, the potential of the others is increased by h/N for some $h > 0$.
- Between spikes, a neuron's potential decays at constant rate, $\dot{u} = -\alpha u$, $\alpha > 0$.
- Parameters : h, α, λ.

A first interesting result

Introduced by Galves, Löcherbach (2013), studied by De Masi, Galves, Löcherbach, Presutti (2015) Fournier, Löcherbach (2016), Robert, Touboul (2016), Duarte, Ost (2016).

Goal : recover, on a toy model, non-trivial large-scale phenomona arising from the interaction

Assumption on λ

$\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous bounded increasing with $\lambda(0) = 0$.

Theorem (Duarte Ost 2016)

If λ is differentiable at 0 then, whatever N, α, h,almost surely, there is a finite number of spikes (i.e. there exists a finite random time after which there is no spike and the system goes to 0).

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$$\mathbb{P}(\text{no spike at all}) = \exp\left(- \sum_{i=1}^{N} \int_0^{+\infty} \lambda\left(e^{-\alpha s} u_i(0)\right) \mathrm{d}s \right) > 0\,.$$

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n_t = population size, death rate = μ birth rate = $(1 - n_t/N)\nu$.

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$$\partial_t n_t \simeq -\mu n_t + (1 - n_t/N)\nu n_t, \qquad \frac{n_t}{N} \to x\,, \qquad \dot{x} = ((1-x)\nu - \mu)x.$$

If $\nu > \mu$, x has a unique globally attractive equilibrium $x_* > 0$.

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- We expect $n_t/N \simeq x_*$ for $1 \ll t \ll e^{\kappa N}$ for some $\kappa > 0$, up to a brutal (unpredictable) extinction : **metastability**.

The Markov process

The generator of the process $U^N(t) = U(t) = (U_i(t))_{i\in[\![1,N]\!]}$, with $U_i(t)$ on $\mathbb{R}_+^N$ is

$$A\varphi(u) \;=\; -\alpha u\cdot\nabla\varphi(u) + \sum_{i=1}^{N}\lambda(u_i)\left(\varphi(u+\Delta_i(u)) - \varphi(u)\right)$$

with

$$(\Delta_i(u))_j = \begin{cases} \frac{h}{N} & j\neq i \\ -u_i & j=i \end{cases},$$

Denote by L^N the last spike time (this is not a stopping time).

Non-linear limit process

Assuming neurons are somewhat independent (propagation of chaos phenomenon) with the same law, for $i \in [\![1, N]\!]$,

$$\frac{h}{N}\sharp\{\text{neuron } j \text{ spikes in } [0,t],\ j \neq i\} \underset{N\to+\infty}{\simeq} h\int_0^t \mathbb{E}\left(\lambda(U_i(s))\right) \mathrm{d}s\,.$$

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Hence, the evolution of a neuron is expected to be similar to the (non-linear/time-inhomogeneous) process $\bar{U}$ with generator

$$\bar{A}_t\varphi(u) \ = \ \left(-\alpha u + h\int_0^{+\infty} \lambda(w)\eta_t(\mathrm{d}w)\right)\cdot\nabla\varphi + \lambda(u)\left(\varphi(0)-\varphi(u)\right)$$

with $\eta_t = \mathcal{L}aw(\bar{U}(t))$.

Interaction with jumps gives, at the limit, a non-linear deterministic drift.

Phase transition

Theorem

1. If $h\lambda'(0) > \alpha$, the non-linear system has at least a non-zero equilibrium ($\eta_t = \mathcal{L}aw(\bar{U}(t)) = \eta_0$ for all $t \geqslant 0$ and $\eta_0 \neq \delta_0$) and there exists $c > 0$ such that for all initial condition $\nu_0 \neq \delta_0$, $\int_0^{+\infty} \lambda(w)\nu_t(\mathrm{d}w) \geqslant c$ for t large enough. The non-zero equilibria have a Lebesgue density of the form
$$g(x) = \frac{p}{hp - \alpha x} \exp\left(- \int_0^x \frac{\lambda(y)}{hp - \alpha y} dy \right) \mathbf{1}_{\{0 \leq x < hp/\alpha\}}$$
with $p = \int_0^\infty \lambda(w) g(x) \mathrm{d}x$.
2. If $h\lambda'(0) < \alpha$ and if λ is concave, the only equilibrium of the non-linear system is δ_0, it is globally attractive in Wasserstein distance :
$$\mathbb{E}\left(\bar{U}(t)\right) \leqslant e^{-(\alpha - h\lambda'(0))t} \mathbb{E}\left(\bar{U}(0)\right) .$$

Exponentially long survival

For simplicity, in the following, we work under the following condition :

Assumption

$\lambda(u) = (ku) \wedge \lambda_*$ for $k, \lambda* > 0$. We write $a = \alpha/(kh)$ and $b = \lambda_*/(kh)$.

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Theorem

Assume $a + b < 1$ and that for all $\varepsilon > 0$, there exists $x_0 > 0$ such that $\mathbb{P}(\sum_{i=1}^N \lambda(U_i^N(0)) \geqslant N x_0) \geqslant 1 - \varepsilon$ for all N large enough (ok if the $U_i(0)$ are i.i.d. non almost surely zero). Then, for all $\delta > 0$,

$$\lim_{N \to +\infty} \mathbb{P}\left(L^N \geqslant e^{(W_0 - \delta)N}\right) \; = \; 1 \,, \qquad \text{with}$$

$$W_0 \; = \; \frac{\lambda_*}{kh}\left(\frac{kh - \lambda_*}{r} - 1 - \ln\left(\frac{kh - \lambda_*}{r}\right) - \frac{1}{2}\ln^2\left(\frac{kh - \lambda_*}{r}\right)\right) > 0 \,.$$

Propagation of chaos

Theorem

Let $U^N = (U_1^N, \ldots, U_N^N)$ be a system of interacting neurons with i.i.d. initial conditions and $\bar{U}^N = (\bar{U}_1^N, \ldots, \bar{U}_N^N)$ independent non-linear processes with $\bar{U}^N(0) = U^N(0)$, so that the jump times of both processes are defined with the same Poisson measures (synchronous coupling). Then, for all $t \geqslant 0$ and $\varepsilon > 0$,

$$\mathbb{E}\left(\sum_{i=1}^N |\bar{U}_i^N(t) - U_i^N(t)|\right) \leqslant h\left(\sqrt{\lambda_* t} + 2t\lambda_*\right) e^{(\alpha+hk+\lambda_*)t}\sqrt{N}$$

$$\mathbb{P}\left(\sup_{s\in[0,t]} \sum_{i=1}^N |U_i^N(s) - \bar{U}_i^N(s)| \geqslant \varepsilon\right) \leqslant \frac{C_t\sqrt{N}}{\varepsilon},$$

with $C_t = 4h(1+\sqrt{\lambda_* t} + t\lambda_*)^2 e^{(2\alpha+hk+\lambda_*)t}$.

Longtime convergence for the non-linear process

For $\nu, \mu \in \mathcal{P}(\mathbb{R}_+)$, consider the Wasserstein distance

$$\mathcal{W}_1(\nu, \mu) = \inf\{\mathbb{E}(|X - Y|), \ X \sim \nu, \ Y \sim \mu\}.$$

Theorem

For a and b small enough (explicit condition), there exists $\kappa \in (0,1)$ such that : for all $\gamma > 0$, there exists $C_\gamma > 0$ such that for all initial conditions μ_0^1, μ_0^2 with $\int_0^\infty \lambda(w)\mu_0^i(w)\mathrm{d}w \geqslant \gamma$, the corresponding solutions of the non-linear system satisfy, for all $t \geqslant 0$,

$$W_1(\mu_t^1, \mu_t^2) \leqslant C_\gamma \kappa^t W_1(\mu_0^1, \mu_0^2).$$

In particular there is a unique equilibrium, globally attractive (if $\mu_0 \neq \delta_0$).

Asymptotic exponentiality

Theorem

Assume a and b are small enough (explicit condition), write $\bar{\lambda}(u) = N^{-1}\sum_{i=1}^{N}\lambda(u_i)$ and $\tau = \inf\{t \geqslant 0,\ U^N \notin \mathcal{D}\}$ with either :

1. $\mathcal{D} = \{\bar{\lambda} \geqslant \gamma\}$ for $\gamma > 0$ small enough (explicit).
2. $\mathcal{D}$ measurable set of $\mathbb{R}_+^N$ so that, for some $\delta > 0$, $\{p_* - \delta \leqslant \bar{\lambda} \leqslant p_* + \delta\} \subset \mathcal{D} \subset \{\bar{\lambda} \geqslant \delta\}$, with $p_* = \int_0^{+\infty}\lambda(w)g(\mathrm{d}w)$ where g is the unique non-linear equilibrium.

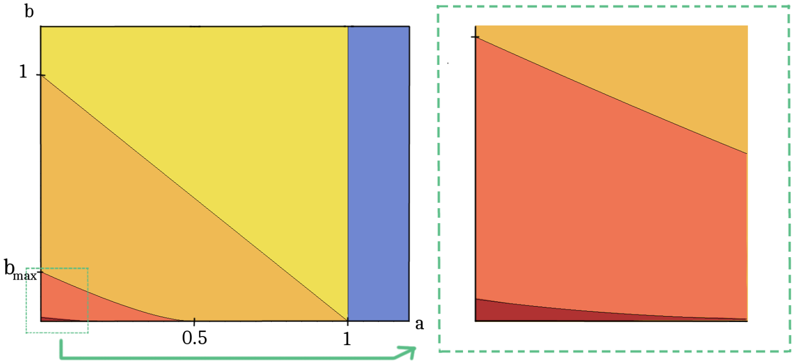
In case (1) we set $\mathcal{K} = \{\bar{\lambda} \geqslant \delta\}$ for some $\delta > \gamma$ and in case (2), $\mathcal{K} = \{p_* - \gamma \leqslant \bar{\lambda} \leqslant p_* + \gamma\}$ for some $\gamma \in (0,\delta)$. Then, in both cases, there exist C, θ, N_0 so that for all $N \geqslant N_0$,

$$\sup_{u,v\in\mathcal{K}}\left|\frac{\mathbb{E}_u(\tau)}{\mathbb{E}_v(\tau)} - 1\right| + \sup_{t\geqslant 0}\sup_{u\in\mathcal{K}}\left|\mathbb{P}_u\left(\tau \geqslant t\mathbb{E}_u(\tau)\right) - e^{-t}\right| \ \leqslant\ \varepsilon(N)$$

with $\varepsilon(N) = Ce^{-\theta N}$ in case (1) and $\varepsilon(N) = C\ln N/N^{1/4}$ in case (2).

General result adapted from Brassesco, Olivieri, Vares (1998).

Summary of the results (except propagation of chaos)



extinction for the limit process

exponentially long survival for a finite size system

exponential exit times for a finite size system

survival for the limit process

global attraction for the limit process

Synchronous coupling

Given a process $(Y_t)_{t\geqslant 0}$ with a jump rate $y\mapsto\lambda(y)$, the jumps of Y can be realized as the jumps of

$$\int_{[0,t]\times\mathbb{R}_+}\mathbf{1}_{z\leqslant\lambda(Y_{s-})}\pi(\mathrm{d}s,\mathrm{d}z)$$

with π a standard Poisson measure on $\mathbb{R}_+\times\mathbb{R}_+$.

Synchronous coupling : two different processes Y and Z are defined with the same Poisson measure. The processes are forced to jump simultaneously as much as possible. Asynchronous jumps occur at rate $|\lambda(Y_t)-\lambda(Z_t)|$.

An auxiliary process

To prove that L_N is exponential with N, we would like to study

$$\Lambda_N(t) \;=\; \frac{1}{N}\sum_{i=1}^{N}\lambda\left(U_i(t)\right)$$

(juste a number, containing all the required information concering jump times). However it is not Markovian.

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- Between jumps,

$$\mathrm{d}\Lambda_N \geqslant -\alpha\Lambda_N$$

- At a jump time,

$$\Lambda_N(t)-\Lambda_N(t-) \;\geqslant\; -\frac{\lambda_*}{N}+\frac{kh}{N}\left(1-\frac{\Lambda_N(t-)}{\lambda_*(1-kh/N)}\right)$$

  (Markov inequality). In particular, if $\lambda_* < kh$ (i.e. $b < 1$), below a given positive threshold, jumps increase Λ_N.
- Jumps occur at rate $N\Lambda_N$.

The process Z_N

Taking the worst case in the evolution of Λ_N :

- Between jumps $\mathrm{d}Z_N = -\alpha Z_N$.
- At a jump,

$$Z_N(t) - Z_N(t-) \ = \ -\frac{\lambda_*}{N} + \frac{kh}{N}\left(1 - \frac{Z_N(t-)}{\lambda_*(1-kh/N)}\right)_+$$

  (more or less).
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Synchronous coupling : $\Lambda_N(t) \geqslant Z_N(t)$ for all $t \geqslant 0$, all jumps of Z_N are jumps of U^N. As $N \to +\infty$, Z_N converges to the solution z of an ODE (with a positive equilibrium, we have large deviations, etc.).

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Besides, using the propagation of chaos result, we get for the non-linear system $\mathbb{E}\left(\lambda(\bar{U}(t))\right) \geqslant z(t)$ (instability of δ_0, and used in the proof of convergence)

Synchronous coupling

Denoting $\bar{\lambda}_t = \mathbb{E}(\lambda(\bar{U}(t))$, consider

$$d\bar{U}(t) = -\alpha\bar{U}(t)dt + h\bar{\lambda}_t dt - \bar{U}(t-)\int_{\mathbb{R}_+} \mathbf{1}_{\{z\leq\lambda(\bar{U}(t-))\}}\pi(dt,dz),$$

and a similar $\hat{U}$ (same π) with a different initial distribution. We want

$$\mathbb{E}\left(|\bar{U}(t)-\hat{U}(t)|\right) \leqslant C\kappa^t, \qquad \kappa\in(0,1).$$

We first show

$$\mathbb{E}\left(\left|\lambda\left(\bar{U}(t)\right)-\lambda\left(\hat{U}(t)\right)\right|\right) \leqslant C\kappa^t, \qquad \kappa\in(0,1),$$

the first point being then similar, using this first information.

Truncated Gronwall

Denoting $f(t) := \mathbb{E}\left(|\lambda(\bar{U}(t)) - \lambda(\hat{U}(t))|\right)$, usually, we would get something like

$$f'(t) \leqslant -\rho f(t) + \varepsilon f(t), \quad \text{i.e.} \quad f(t) \leqslant f(0) + (\varepsilon - \rho) \int_0^t f(s)\mathrm{d}s$$

where ε is small if the non-linearity is small. In fact in our case we get

$$f(t) \ \leqslant \ \theta \int_{(t-t_*)_+}^t f(s)\mathrm{d}s + \left(kh \int_0^t f(s)\mathrm{d}s + f(0) \right) \mathbf{1}_{t\in[0,t_*]}$$

with $\theta > 0$ and t_* is (more or less) the time from 0 to saturation of

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Iterating Gronwall's Lemma, for $n \in \mathbb{N}$ and $t \in [nt_*, (n+1)t_*]$,

$$f(t) \ \leqslant \ (\theta t_* e^{\theta t_*})^n e^{(\theta+kh)t_*} f(0)\,.$$

For a, b small enough, θt_* is small enough to get a contraction.

Conclusion

- Can we prove asymptotic exponentiality for L^N ?
- Is it possible to have non-uniqueness of the positive equilibrium for the non-linear system ?
- Could the quantitative bounds established for proving the asymptotic exponentiality be used to obtain convergence of the process conditioned not to have left $\mathcal{D}$ toward the quasi-stationary distribution (at rate independent from N, or at least polynomial in N) ?

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Thanks for your attention !