Quelques problèmes d'optimisation sur les bobines de stellarators.

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Nuclear fusion confinement

- Goal : Confine a plasma of approx. 150 millions K for as long as possible with a density as high as possible in order to achieve fusion ignition.
- Solution : A plasma is made of ionized particules, thus interacts with a magnetic field.

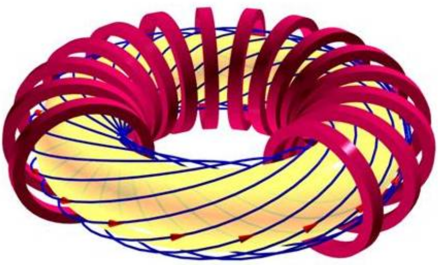

Figure: magnetic field lines inside a Tokamac, Inria team TONUS

Stellarators

Stellarator approach : The magnetic confinement relies mainly on external coils.

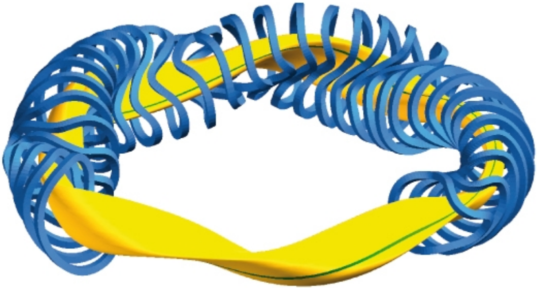

Figure: Wendelstein 7-X, Max-Planck Institut für Plasmaphysik

The plasma shape and the coils are obtained by several optimizations.

Typical approach

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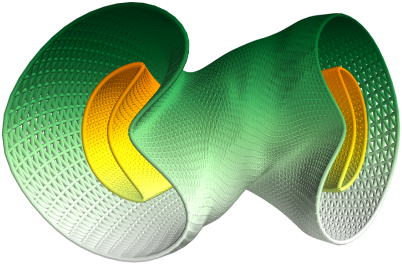

Figure: Coil winding surface and plasma surface of the NCSX Stellarator.

Typical approach

1. Find a good magnetic field to ensure the plasma confinement.
2. We use a 'Coil winding surface' and find a current-sheet to generate the given B_{target}.
3. (Approximate the current-sheet by several coils)

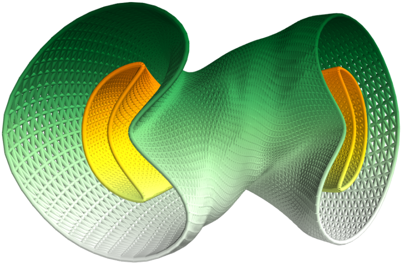

Figure: Coil winding surface and plasma surface of the NCSX Stellarator.

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The magnetic field generated by the electric currents on the CWS (denoted S).

Biot-Savart law in vacuo

$$\forall y \notin S, B(y) = \mathrm{BS}(j)(y) = \int_S j(x) \times \frac{y-x}{|y-x|^3} dS(x), \tag{1}$$

The figure of merit we use to ensure $B \approx B_{target}$ is

plasma-shape objective

$$\chi_B^2(j) = \int_P |\mathrm{BS}(j)(y) - B_{target}(y)|^2 dy \tag{2}$$

The goal

$$\inf_{\substack{j \in L^2(\mathfrak{X}(S)) \\ \operatorname{div} j = 0}} \chi_B^2(j) \tag{3}$$

An inverse problem

$BS(\cdot)$ is continuous from $L^2(\mathfrak{X}(S)) \to C^k(\partial P, \mathbb{R}^3)$
$\implies j \mapsto BS(j)$ is compact (from $L^2(\mathfrak{X}(S)) \to L^2(P, \mathbb{R}^3)$).

- Use a finite dimensional subspace [2].
- Use a Tychonoff regularization [1].

$$\chi_j^2 = \int_S |j|^2 dS. \tag{4}$$

Lemma

For any $\lambda > 0$, the problem

$$\inf_{\substack{j \in L^2(\mathfrak{X}(S)) \\ \operatorname{div} j = 0}} \chi_B^2 + \lambda \chi_j^2 \tag{P}$$

admits a unique minimizer.

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Motivations

- Building a Stellarator is expensive. . .
- compact Stellarators require higher magnetic field
- Higher magnetic fields call for higher currents
- $\implies$ The Laplace forces ($d\vec{F} = i\vec{dl} \wedge \vec{B}$) grew quadratically.

$\implies$ The Laplace forces must be optimized.

Problem

How can we define the Laplace forces on a current-sheet?

Statement of the problem

Let S a toroidal surface and $j \in \mathfrak{X}(S)$ a vector field.

Biot and Savart

$$\forall y \notin S, B(y) = \mathrm{BS}(j)(y) = \int_S j(x) \times \frac{y-x}{|y-x|^3} dS(x),$$

Not integrable

B is not defined on S, indeed for any $y \in S$,

$$\int_S \frac{1}{|x-y|^2} dx = \infty$$

There is a magnetic discontinuity on the surface given by

$$B^1_T - B^2_T = n_{12} \wedge j.$$

About the Laplace forces

- B does not blow up near S.
- The discontinuity of B is responsible for a normal force proportional to $|j|^2$ trying increase the thickness of S.

Average Laplace forces

We focus on the other contributions of the Laplace forces, and therefore we define:

$$L_\varepsilon(j)(y) = \frac{1}{2}\left(j \wedge \left[B(j)(y + \varepsilon n(y)) + B(j)(y - \varepsilon n(y))\right]\right)$$

$$L(j) = \lim_{\varepsilon \to 0} L_\varepsilon(j)$$

This definition raises several questions:

1. Under which assumptions on j can we ensure that $L(j)$ is well defined?
2. Can we find an explicit expression of $L(j)$ (i.e. without a limit on ε)?
3. Which functional space does $L(j)$ belong to (for j in a given functional space)?

A 3 scales problem

To compute L from L_ε, we need 3 scales :

1. the discretisation-length of S : h,
2. the infinitesimal displacement ε,
3. the characteristic distance of variation of the magnetic field, d_B.

With :

- $h \ll \varepsilon$ as $\int_S |y + \varepsilon n(y) - x|^{-2} dS(x)$ blows up when $\varepsilon \to 0$.
- $\varepsilon \ll d_B$ to approximate L.

Theorem

Theorem [3]

Suppose $j_1, j_2 \in \mathfrak{X}^{1,2}(S)$, then $L_\varepsilon(j_1, j_2)$ has a limit in $L^p(S, \mathbb{R}^3)$ for any $1 \leq p < \infty$ when $\varepsilon \to 0$, denoted $L(j_1, j_2)$. Besides, L is a continuous bilinear map $\mathfrak{X}^{1,2}(S) \times \mathfrak{X}^{1,2}(S) \to L^p(S, \mathbb{R}^3)$ given by

$$L(j_1, j_2)(y) = -\int_S \frac{1}{|y-x|}\left[\operatorname{div}_x(\pi_x j_1(y)) + \pi_x j_1(y) \cdot \nabla_x\right] j_2(x)dx \tag{5}$$

$$+ \int_S \langle j_1(y) \cdot n(x)\rangle \frac{\langle y-x, n(x)\rangle}{|y-x|^3} j_2(x)dx \tag{6}$$

$$+ \int_S \frac{1}{|y-x|}\left[\langle j_1(y) \cdot j_2(x)\rangle \operatorname{div}_x(\pi_x) + \nabla_x \langle j_1(y) \cdot j_2(x)\rangle\right] dx \tag{7}$$

$$- \int_S \langle j_1(y) \cdot j_2(x)\rangle \frac{\langle y-x, n(x)\rangle}{|y-x|^3} n(x)dx \tag{8}$$

Ideas of the proof

- Note that $\frac{y-x}{|y-x|^3} = -\nabla_x \frac{1}{|y-x|}$.
- Do an integration by part on the tangential component of the gradient.
- Use some estimates when ε is small to eliminate the part responsible for the magnetic discontinuity.

Tools : Hardy-Littlewood-Sobolev inequality and Sobolev embeding on compact manifold.

Optimization

We introduce the following costs:

- χ_B to ensure that we produce the magnetic field chosen :

$$\chi_B^2 = \int_{\partial P} \langle B(x) \cdot n(x) \rangle^2 dS(x)$$

- A penalization term on j

$$\chi_j^2 = \int_S |j|^2 dS$$

$$\chi_{\nabla j}^2 = \int_S (|\nabla j_x|^2 + |\nabla j_y|^2 + |\nabla j_z|^2) dS.$$

- A penalizing term on the Laplace forces, for example $L^p(S, \mathbb{R}^3)$

$$|L(j)|_{L^p} = \Big(\int_S |L(j)|_2^p \Big)^{1/p} dS$$

Thus, we will minimize the new cost with relative weights $\lambda_1, \lambda_2, \gamma \geq 0$.

$$\chi^2 = \chi_B^2 + \lambda_1 \chi_j^2 + \lambda_2 \chi_{\nabla j}^2 + \gamma |L(j)|_{L^p}$$

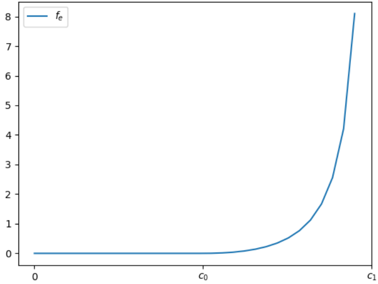
Lemma

Suppose $\lambda_1, \lambda_2, \gamma > 0$ *and* $p < \infty$ *then*

$$\inf_{j \in E} \chi_B^2 + \lambda_1 \chi_j^2 + \lambda_2 \chi_{\nabla j}^2 + \gamma |L(j)|_{L^p}$$

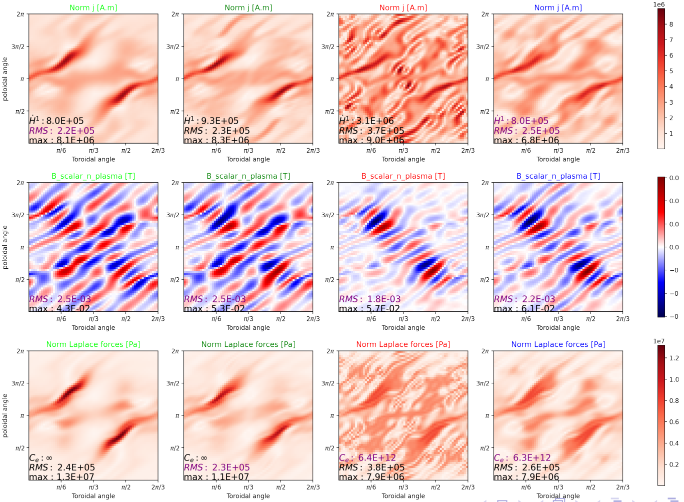
admit a minimizer.

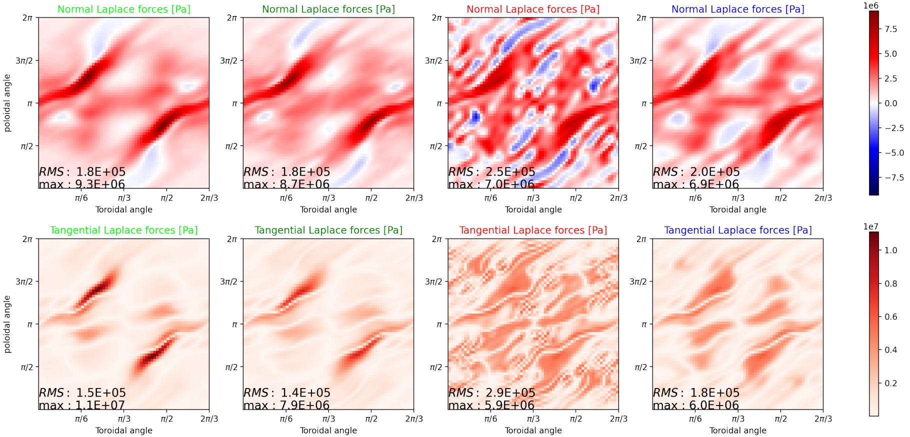
We also introduce a cost to penalize only high values of the forces:
$C_e = \int_S f_e(|L(j)|)$

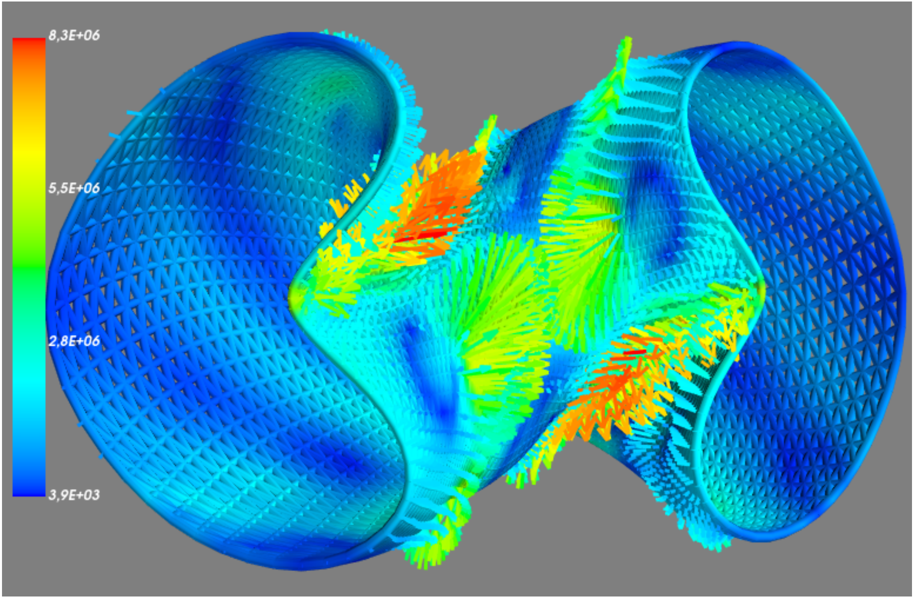


Case	λ_1 ($\mathrm{T}^2\ \mathrm{m}^2/\mathrm{A}^2$)	λ_2 ($\mathrm{T}^2\ \mathrm{m}^4/\mathrm{A}^2$)	γ ($\mathrm{T}^2/\mathrm{Pa}^2$)	χ^2_F
1	$1.5 \cdot 10^{-16}$	0	0	0
2	0	0	10^{-17}	$\lvert L(j)\rvert^2_{L^2(S,\mathbb{R}^3)}$
3	0	0	10^{-16}	C_e
4	10^{-19}	10^{-19}	10^{-16}	C_e

(9)




Normal Laplace forces [Pa]
RMS : 1.8E+05
max : 9.3E+06
Normal Laplace forces [Pa]
RMS : 1.8E+05
max : 8.7E+06
Normal Laplace forces [Pa]
RMS : 2.5E+05
max : 7.0E+06
Normal Laplace forces [Pa]
RMS : 2.0E+05
max : 6.9E+06
1e6
Tangential Laplace forces [Pa]
RMS : 1.5E+05
max : 1.1E+07
Tangential Laplace forces [Pa]
RMS : 1.4E+05
max : 7.9E+06
Tangential Laplace forces [Pa]
RMS : 2.9E+05
max : 5.9E+06
Tangential Laplace forces [Pa]
RMS : 1.8E+05
max : 6.0E+06
1e7
poloidal angle
Toroidal angle


8,3E+06
5,5E+06
2,8E+06
3,9E+03

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We want to optimize on both the current sheet and the Coil Winding Surface.

Admissible shapes

- Topology of a torus
- Regular enough
- Far enough to the plasma

Shape optimization problem

$$\inf_{S \text{ admissible}} \left(\inf_{j \in L^2_0(\mathfrak{X}(S))} \chi_B^2 + \lambda \chi_j^2 \right) \tag{SOP}$$

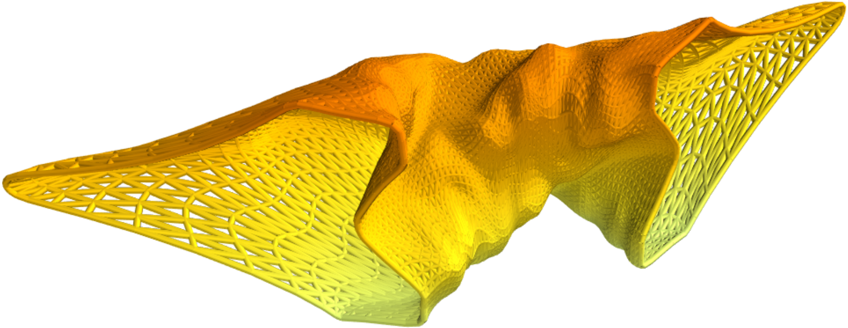
Some preliminary numerical results : $\lambda = 2.5e^{-16}$

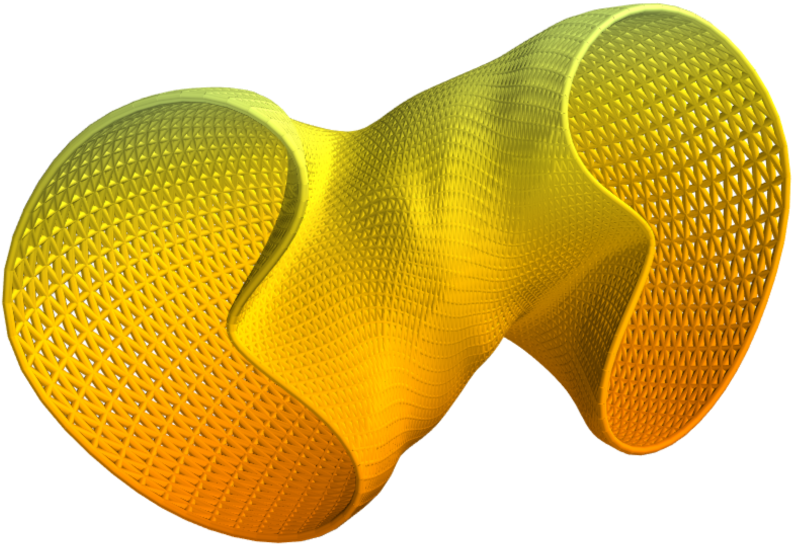
Costs

Name	χ_B^2	$\lvert B_{err}\rvert_\infty$	χ_j^2	$\lvert j\rvert_\infty$	*EMcost*
ref	$4.80 \cdot 10^{-3}$	$5.15 \cdot 10^{-2}$	$1.43 \cdot 10^{14}$	$7.42 \cdot 10^{6}$	$4.06 \cdot 10^{-2}$
DPC	$1.23 \cdot 10^{-3}$	$3.20 \cdot 10^{-2}$	$9.48 \cdot 10^{13}$	$5.99 \cdot 10^{6}$	$2.49 \cdot 10^{-2}$

Geometry

Name	Distance (m)	Perimeter (m^2)	Maximal curvature (m^{-1})
Ref	$1.92 \cdot 10^{-1}$	$5.57 \cdot 10^{1}$	$1.19 \cdot 10^{1}$
DPC	$1.99 \cdot 10^{-1}$	$5.60 \cdot 10^{1}$	$1.30 \cdot 10^{1}$





Prospects

- Proof of existence of a solution to the shape optimisation problem,
- Collaboration with Renaissance fusion for industrial applications,
- Laplace forces in the shape optimisation.

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