Sparse moment-sum-of-squares relaxations of nonlinear dynamical systems with guaranteed convergence

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Il y a la système dynamique $\dot{x} = f(x)$ avec $X \subset \mathbb{R}^n$ compact. Les solutions s’appellent $\varphi_t(\cdot)$.

**Definition (Maximum positively invariant (MPI) set)**

Le set de tous les points $x_0 \in X$ tel que $\varphi_t(x_0) \in X$ pour tous les temps $t \in \mathbb{R}_+$, s’appelle MPI set.
Given a dynamical system \( \dot{x} = f(x) \) on \( X \subset \mathbb{R}^n \) compact we denote the solution/flow map by \( \varphi_t(\cdot) \).
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**Definition (Maximum positively invariant (MPI) set)**

Set of initial conditions $x_0 \in X$, such that the solutions $\varphi_t(x_0)$ stay in $X$ for all $t \in \mathbb{R}_+$, is called the MPI set.
In Korda et al. (2014) the following linear program for computing the MPI set is proposed

\[ d^* := \inf_{\lambda} \int_{\mathcal{X}} w(x) \, d\lambda(x) \]

s.t. 
\[ v \in C^1(\mathbb{R}^n), \, w \in C(\mathcal{X}) \]
\[ \beta v(x) - \nabla v \cdot f(x) \geq 0 \quad \text{for } x \in \mathcal{X} \]
\[ w \geq 0 \]
\[ w - v - 1 \geq 0 \]
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**Theorem**

It holds $$d^* = \lambda(\text{MPI})$$ and $$w^{-1}([1, \infty)) \supset \text{MPI}$$ for all feasible $$(v, w)$$. 
The corresponding hierarchy of SDPs

The corresponding semidefinite programs (SDPs) have the following form

\[
d^*_k := \inf \int_X w(x) \, d\lambda(x)
\]

s.t.

\[
\beta v - \nabla v \cdot f = \text{SoS}_k
\]

\[
v, w \in \mathbb{R}[x]_k
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*It holds* \( d_k^* \to d^* = \lambda(\text{MPI}) \) *and* \( w_k^{-1}([1, \infty)) \supset \text{MPI} \) *for all feasible* \((v_k, w_k)\).
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Also other methods typically suffer from the curse of dimensionality.

\[\leadsto \text{reduction techniques are needed to solve larger problems}\]
The sparsity graph of $f$ represents the dependence of different states in the dynamics in the following way.

[Figure: Example of a sparsity graph]

where the nodes $x_1, x_3, x_4$ represent states in $\mathbb{R}^2$, $x_2$ represents a scalar state and $x_3$ a state in $\mathbb{R}^3$. The arrows indicate

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1), \\
\dot{x}_2 &= f_2(x_1, x_2), \\
\dot{x}_3 &= f_3(x_1, x_3, x_4), \\
\dot{x}_4 &= f_4(x_1, x_4), \\
\dot{x}_5 &= f_5(x_2, x_4).
\end{align*}
\]
Let $x_1, \ldots, x_k$ be the (grouped) states and $f = (f_1, \ldots, f_k)$ then there is an edge between the nodes $x_i$ and $x_j$ if $f_j$ depends on $x_i$.

![Sparsity graph](image)

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\dot{x}_4 &= f_4(x_1, x_4), \\
\dot{x}_5 &= f_5(x_2, x_4).
\end{align*}
\]
We will replace the large SDP by a collection of smaller SDPs. The size of the largest among these depends on the largest weighted past of the graph.

Figure: Largest weighted past of the example graph
The main theorem

Theorem

There exists a convergent hierarchy of sum-of-squares problems with the largest sum-of-squares multiplier containing $\omega$ variables, where $\omega$ is the longest weighted past in the graph.
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1. Sparse description of the sets of interest
2. Formulating and solving the corresponding linear optimization problems
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Theorem

There exists a convergent hierarchy of sum-of-squares problems with the largest sum-of-squares multiplier containing $\omega$ variables, where $\omega$ is the longest weighted past in the graph.

1. Sparse description of the sets of interest
2. Formulating and solving the corresponding linear optimization problems
3. “Glue” the results together to obtain (an approximation of) the MPI set.
The most basic sparsity structure that allows a reduction is the following “cherry structure” from Chen et al. (2018):

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1) \\ 
\dot{x}_2 &= f_2(x_1, x_2) \\ 
\dot{x}_3 &= f_3(x_1, x_3)
\end{align*}
\]

Hence they induce the (almost) independent subsystems

\[
\begin{align*}
\dot{(x_1, x_2)} &= (f_1, f_2)(x_1, x_2) \\ 
\dot{(x_1, x_3)} &= (f_1, f_3)(x_1, x_3) \\ 
\dot{x}_1 &= f_1(x_1)
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Prototype sparse structure

The most basic sparsity structure that allows a reduction is the following “cherry structure” from Chen et al. (2018)

\[ \dot{x}_1 = f_1(x_1) \text{ on } \mathbb{R}^{n_1} \]
\[ \dot{x}_2 = f_2(x_1, x_2) \text{ on } \mathbb{R}^{n_2} \]
\[ \dot{x}_3 = f_3(x_1, x_3) \text{ on } \mathbb{R}^{n_3} \]
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Hence they induce the (almost) independent subsystems

\[ (\dot{x}_1, \dot{x}_2) = (f_1, f_2)(x_1, x_2), \quad (\dot{x}_1, \dot{x}_3) = (f_1, f_3)(x_1, x_3), \quad \dot{x}_1 = f(x_1) \]
Assumption: Sparsity in the constraint set – $X = X_1 \times X_2 \times X_3$
Sparse description of the MPI set

**Assumption:** Sparsity in the constraint set – \( X = X_1 \times X_2 \times X_3 \)

**Lemma (Gluing)**

Let \( X = X_1 \times X_2 \times X_3 \) and the dynamical system be sparse (in the sense as above) and let \( M_1^+ \) and \( M_2^+ \) denote the MPI sets for the subsystems on \((x_1, x_2)\) and \((x_1, x_3)\) then the MPI set \( M_+ \) of the whole system is given by

\[
\{(x_1, x_2, x_3) \in X : (x_1, x_2) \in M_1^+, (x_1, x_3) \in M_2^+\}.
\]

**Proof.**

Uses that the dynamical system decouples into subsystems (the state constraint decoupling is essential as well).
Sparse description of the MPI set

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Lemma (Gluing)

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$$\{ (x_1, x_2, x_3) \in X : (x_1, x_2) \in M_+^1, (x_1, x_3) \in M_+^2 \}.$$

Proof.

Uses that the dynamical system decouples into subsystems (the state constraint decoupling is essential as well).

This allows to solve separate SDPs for the subsystems instead of solving a SDP for the whole system.
Convergence

Proposition

For $i = 1, 2$ let $M^i$ denote the MPI set for the two subsystems on $(x_1, x_2)$ and $(x_1, x_3)$ and $M$ denotes the MPI set for the whole system. Let $M^i \subset M^i_k \subset X_1 \times X_{i+1} \subset \mathbb{R}^{n_1+n_{i+1}}$ for $i = 1, 2$ and

$$M_{k}^{1,2} := \{(x_1, x_2, x_3) \mid (x_1, x_2) \in M_k^1, (x_1, x_3) \in M_k^2\}.$$ 

Then

$$\lambda(M_{k}^{1,2} \setminus M) \leq \lambda(M_k^1 \setminus M^1)\lambda(X_3) + \lambda(M_k^2 \setminus M^2)\lambda(X_2). \quad (1)$$

In particular if $M^i_k$ converges to $M^i$ with respect to $\lambda$ for $i = 1, 2$ then $M_{k}^{1,2}$ converges to $M$ with respect to $\lambda$. 
General procedure

Given a dynamical system induced by $f$ and a method for approximating/computing the region of attraction, MPI set or (global) attractors for an arbitrary dynamical system.

i. Find the minimal subsystems that cover the whole system.

ii. Compute (outer) approximations $S_i$ for these subsystems.

iii. Glue the set $S_i$ together as in the gluing lemma.
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**Theorem (Main theorem)**

This procedure produces convergent approximations of the desired sets if the sets $S_i$ are converging approximations of the desired sets for the subsystems. In case of the sum-of-squares approach we have convergence w.r.t Lebesgue measure discrepancy and the size of the largest occurring SDP depends only on the largest weighted pasts of the sparsity graph of $f$. 
Remark

We call a set of states \((x_i)_{i \in I}\) for some index set \(I \subset \{1, \ldots, n\}\) a subsystem of \(\dot{x} = f(x)\) if we have

\[
f_I \circ P_I = P_I \circ f
\]

where \(f_I := (f_i)_{i \in I}\) denotes the components of \(f\) according to the index set \(I\) and \(P_I : \mathbb{R}^n \rightarrow \mathbb{R}^{|I|}\) denotes the canonical projection onto the states \(x_I\), i.e. \(P_I(x) := (x_i)_{i \in I}\).
Finding a optimal decomposition

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Remark

Finding an optimal decomposition is only based on the sparsity graph of \(f\) and the product structure of \(X\). First decompose \(X = X_1 \times \ldots \times X_k\) (up to permutation of coordinates) and construct the sparsity graph of \(f\) with respect to the states \(x_i \in X_i\). The minimal subsystems are characterized by the resulting sparsity graph and can be found fast using well known algorithms.
A hard example

The graph of the dynamics considered in Tacchi et al. (2019) is given by

\[ x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow \ldots \rightarrow x_r \]

Our approach does not lead to a reduction for this example.
For a Van der Pol oscillator with cherry structure of the form
We get

for degree $k = 8$ and $N = 9$ and total dimension 20
Numerical example

and

for degree $k = 8$ and $N = 25$ and total dimension 52.
Conclusion

Contribution of this work

1. Can be applied in a similar way for the region of attraction and global attractors
2. First sparse method to approximate the MPI set, global attractors and the region of attraction with guaranteed convergence
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Outlook and perspectives

1. Extending to sparse control systems.
2. Improve/increase exploiting sparse structures.
3. Exploit polynomial structure of $f$ and not only focusing sparse coupling; as for example term sparsity.
4. Coordinate free formulation.