Numerical schemes for the hyperbolic Shallow-Water type model with two velocities

Nelly BOULOS AL MAKARY
N. AGUILLON, E. AUDUSSE, M.PARISOT

LAGA, Université Paris13
Fondation Sciences Mathématiques de paris
LJLL, Sorbonne Université
ANGE, INRIA Paris

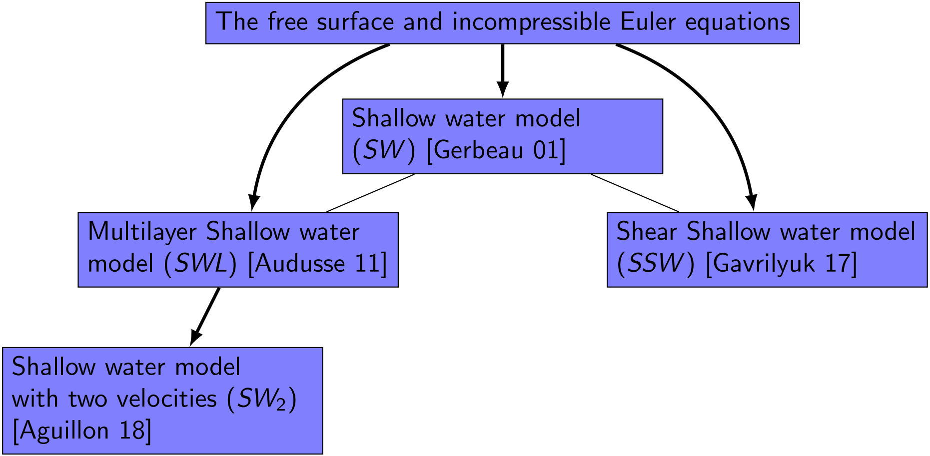
This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 754362

Motivation

FIGURE – Floods in Vésubie, France 2020

FIGURE – Sediment transport and Deposition of the Rhone River

Motivation


The free surface and incompressible Euler equations
Shallow water model (SW) [Gerbeau 01]
Multilayer Shallow water model (SWL) [Audusse 11]
Shear Shallow water model (SSW) [Gavrilyuk 17]
Shallow water model with two velocities (SW_2) [Aguillon 18]

Contents

Introduction

The $2D$ shallow water type model with two velocities reads

$$\left\{\begin{array}{lll} \partial_t h & + \nabla \cdot (h\overline{U}) & = 0, \\ \partial_t (h\overline{U}) & + \nabla \cdot (h(\overline{U} \otimes \overline{U} + \hat{U} \otimes \hat{U}) + \frac{g}{2} h^2 \mathbb{I}) & = 0, \\ \partial_t \hat{U} & + (\overline{U} \cdot \nabla)\hat{U} + (\hat{U} \cdot \nabla)\overline{U} & = 0. \end{array}\right. \qquad (SW_2)$$

The system is invariant by rotation so we study (SW_2) in the x-direction

$$\left.\begin{array}{lll} \partial_t h & + \partial_x (h\overline{u}) & = 0 \\ \partial_t (h\overline{u}) & + \partial_x (h(\overline{u}^2 + \hat{u}^2) + \frac{g}{2} h^2) & = 0 \\ \partial_t \hat{u} & + \partial_x (\overline{u}\hat{u}) & = 0 \end{array}\right\} \qquad (1D)$$

$$\left.\begin{array}{lll} \partial_t (h\overline{v}) & + \partial_x (h(\overline{uv} + \hat{u}\hat{v})) & = 0 \\ \partial_t \hat{v} & + \hat{u}\partial_x \overline{v} + \overline{u}\partial_x \hat{v} & = 0 \end{array}\right\}$$

with $\quad \overline{U} = (\overline{u}, \overline{v})^t \quad$ and $\quad \hat{U} = (\hat{u}, \hat{v})^t.$

Introduction

We consider the set of variables

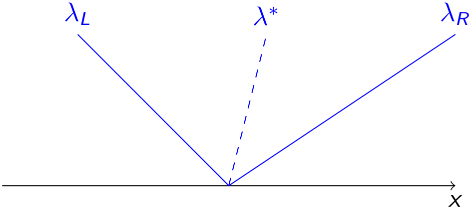
$$U = (h, \overline{u}, \hat{u})$$

and the initial data for the Riemann problem

$$U(0,x) = \begin{cases} U_L = (h_L, \overline{u}_L, \hat{u}_L)^t \in \mathbb{R}_+^* \times \mathbb{R}^2 & \text{if } x < 0, \\ U_R = (h_R, \overline{u}_R, \hat{u}_R)^t \in \mathbb{R}_+^* \times \mathbb{R}^2 & \text{if } x \geq 0. \end{cases}$$

The eigenvalues are given by

$$\begin{cases} \lambda_L &= \overline{u} - \sqrt{gh + 3\hat{u}^2} \\ \lambda^* &= \overline{u}, \\ \lambda_R &= \overline{u} + \sqrt{gh + 3\hat{u}^2}, \end{cases}$$



Description of the Godunov-type schemes

Finite volume framework

- We consider a uniform discretization of the computational domain
- We denote $C_i =]x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}[$ the cell of length $\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ and centered at x_i
- For any time t^n, we define $t^{n+1} = t^n + \Delta t^n$ with Δt^n satisfying a CFL condition to be described later
- Let U_i^n be a piecewise constant approximation of $U(x,t)$ at time t^n on the cell C_i
- We propose the following update formula

$$\forall i \in \mathbb{Z}, \forall n \in \mathbb{N} \quad U_i^{n+1} = U_i^n - \frac{\Delta t^n}{\Delta x}\left(\mathcal{F}_{i+\frac{1}{2}}^n - \mathcal{F}_{i-\frac{1}{2}}^n\right),$$

where

$$\mathcal{F}_{i+\frac{1}{2}}^n \approx \frac{1}{\Delta t^n}\int_{t^n}^{t^{n+1}} F\left(U\left(t, x_{i+\frac{1}{2}}\right)\right) dt.$$

The initialization of the algorithm can be computed with

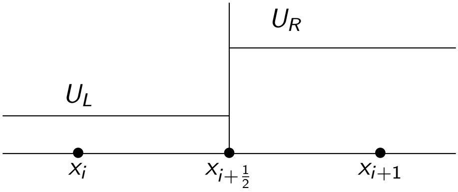
$$\forall i \in \mathbb{Z} \quad U_i^0 = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} U(x,0) dx.$$

Description of the Godunov-type schemes

Godunov method

Godunov observed that U_i^n define at each cell interface $x_{i+\frac{1}{2}}$ a Riemann problem

$$\begin{cases} \partial_t U + \partial_x F(U) &= 0 \\ U(t^n, x) &= \begin{cases} U_L & \text{if } x < x_{i+\frac{1}{2}} \\ U_R & \text{if } x \geq x_{i+\frac{1}{2}} \end{cases} \end{cases}$$



Description of the Godunov-type schemes

The considered approximated Riemann solvers are

$$\tilde{U}\left(\frac{x}{t}, U_L, U_R\right) = \begin{cases} U_L = U_{\frac{1}{2}} & \text{if } \frac{x}{t} < \lambda_1, \\ \tilde{U}_{j+\frac{1}{2}} & \text{if } \lambda_j < \frac{x}{t} < \lambda_{j+1} \text{ for } j = 1, ..., N-1, \\ U_R = U_{N+\frac{1}{2}} & \text{if } \frac{x}{t} > \lambda_N. \end{cases}$$

The update at time t^{n+1} is then defined by

$$U_i^{n+1} = \frac{1}{\Delta x} \int_0^{\frac{\Delta x}{2}} \tilde{U}\left(\frac{x}{\Delta t^n}, U_{i-1}^n, U_i^n\right) dx + \frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^0 \tilde{U}\left(\frac{x}{\Delta t^n}, U_i^n, U_{i+1}^n\right) dx.$$

Description of the Godunov-type schemes

1. The external waves of the approximated solution have to be faster than the external wave speed of the exact solution

$$\begin{aligned} \lambda_L &= \min\left(\overline{u}_L - c_L, \overline{u}_R - c_R\right), \\ \lambda_R &= \max\left(\overline{u}_L + c_L, \overline{u}_R + c_R\right), \end{aligned}$$

   where $c_X = \sqrt{gh_X + 3\hat{u}_X^2}$.

2. The time step has to satisfy the following CFL condition

$$(\max_{j,i} |\lambda^n_{j,i+\frac{1}{2}}|)\Delta t^n \leq \frac{\Delta x}{2},$$

   where $\lambda^n_{j,i+\frac{1}{2}} = \lambda_j(U_i^n, U_{i+1}^n)$.

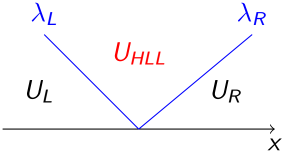
3. The scheme has to satisfy a consistency property in the sense Harten and Lax showed in (Lax 83)

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \tilde{U}(\frac{x}{\Delta t^n}, U_L U_R)dx = \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} U_r(\frac{x}{\Delta t^n}, U_L, U_R)dx,$$

Homogenuous 1D Shallow water model with two velocities

The *HLL* approximate Riemann solver

$$\tilde{U}_{HLL}\left(\frac{x}{t}, U_L, U_R\right) = \begin{cases} U_L & \text{if } \frac{x}{t} < \lambda_L, \\ U_{HLL} & \text{if } \lambda_L < \frac{x}{t} < \lambda_R, \\ U_R & \text{if } \frac{x}{t} > \lambda_R, \end{cases}$$



The consistency with the integral form of the conservation law leads to the following intermediate states

$$\begin{cases} h_{HLL} &= \dfrac{[h\,(\lambda - \overline{u})]}{[\lambda]}, \\ h_{HLL}\overline{u}_{HLL} &= \dfrac{\left[\lambda h\overline{u} - h(\overline{u}^2 + \hat{u}^2) - \frac{g}{2}h^2\right]}{[\lambda]}, \\ \hat{u}_{HLL} &= \dfrac{[\hat{u}\,(\lambda - \overline{u})]}{[\lambda]}. \end{cases}$$

Assume $h_L > 0$ or $h_R > 0$ then, the intermediate state h_{HLL} is positive.

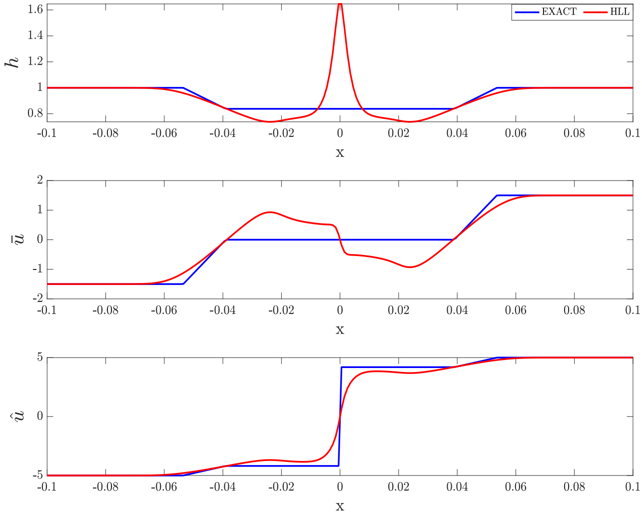


FIGURE – Two rarefactions case. Plots of the variables using *HLL* solver for 1000 grid cells

1D Shallow water model with two velocities

The HLL^* scheme

- The quantity $\frac{\hat{u}}{h}$ jumps only along the intermediate contact discontinuity. In fact, from (SW_2), we can deduce that for regular solutions

$$\partial_t(\frac{\hat{u}}{h}) + \overline{u}\partial_x(\frac{\hat{u}}{h}) = 0.$$

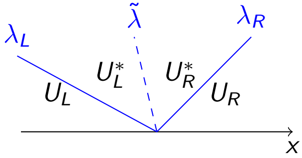
- The HLL scheme is used to update only the classical shallow water variables $(h, \overline{u})$ and to compute the interface mass and momentum fluxes $\mathcal{F}^h_{HLL}$ and $\mathcal{F}^{h\overline{u}}_{HLL}$.
- The shear velocity $\hat{u}$ is updated using an upwind strategy

$$\mathcal{F}^{\hat{u}}_{\{HLL,up,i+\frac{1}{2}\}} = \frac{\hat{u}^n_i}{h^n_i}\mathcal{F}^{h+}_{\{HLL,i+\frac{1}{2}\}} + \frac{\hat{u}^n_{i+1}}{h^n_{i+1}}\mathcal{F}^{h-}_{\{HLL,i+\frac{1}{2}\}},$$

1D Shallow water model with two velocities

The HLL^* solver can also be seen as a $3-$ waves approximate Riemman solver

$$\tilde{U}_{HLL^*}(\frac{x}{t}, U_L, U_R) = \begin{cases} U_L & \text{if } \frac{x}{t} < \lambda_L, \\ U_L^* & \text{if } \lambda_L < \frac{x}{t} < \tilde{\lambda}, \\ U_R^* & \text{if } \tilde{\lambda} < \frac{x}{t} < \lambda_R, \\ U_R & \text{if } \frac{x}{t} > \lambda_R. \end{cases}$$



To construct the numerical scheme

1. we consider the consistency relations
2. we impose the continuity of h and $\overline{u}$ through the $\tilde{\lambda}-$wave

$$h_L^* = h_R^* \qquad \text{and} \qquad \overline{u}_L^* = \overline{u}_R^*.$$

3. we impose the continuity of $\dfrac{\hat{u}}{h}$ on the external waves λ_L and λ_R

$$\frac{\hat{u}_L}{h_L} = \frac{\hat{u}_L^*}{h_L^*} \qquad \text{and} \qquad \frac{\hat{u}_R}{h_R} = \frac{\hat{u}_R^*}{h_R^*}.$$

1D Shallow water model with two velocities

For $\frac{\hat{u}_R}{h_R} \neq \frac{\hat{u}_L}{h_L}$, the intermediate states are

$$\begin{cases} h_L^* = h_R^* = h_{HLL}, \\ \overline{u}_L^* = \overline{u}_R^* = \overline{u}_{HLL}, \\ \hat{u}_L^* = \hat{u}_L \dfrac{h_{HLL}}{h_L}, \\ \hat{u}_R^* = \hat{u}_R \dfrac{h_{HLL}}{h_R}, \\ \tilde{\lambda} = \lambda_R - \dfrac{h_R \left(\lambda_R - \overline{u}_R\right)}{h_{HLL}}. \end{cases} \tag{HLL*}$$

If $\frac{\hat{u}_R}{h_R} = \frac{\hat{u}_L}{h_L}$, we get $\hat{u}_L^* = \hat{u}_R^* = \hat{u}_{HLL}$ and we choose $\tilde{\lambda}$ defined above.
In addition, we have

- $\lambda_L < \tilde{\lambda} < \lambda_R$
- $sgn(\mathcal{F}^h_{\{HLL\}}) = sgn(\tilde{\lambda})$
- $\mathcal{F}^{\hat{u}}_{\{HLL,up\}} = \mathcal{F}^{\hat{u}}_{HLL^*}$

Homogenuous 1D Shallow water model with two velocities

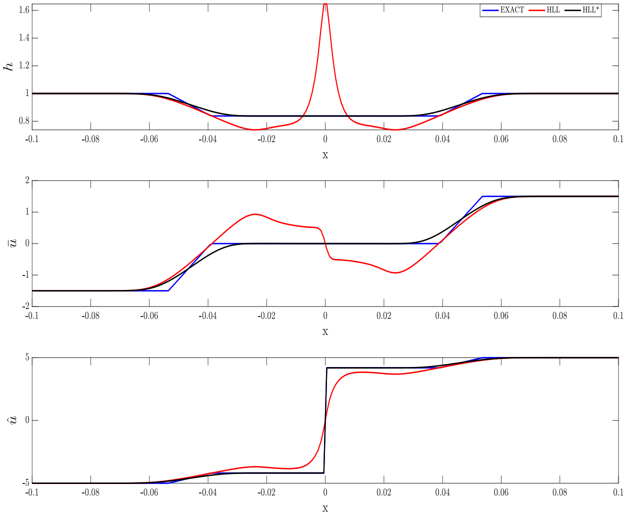


FIGURE – Two rarefactions case. Plots of the variables using *HLL* and HLL^* solvers for 1000 grid cells

Homogenuous 1D Shallow water model with two velocities

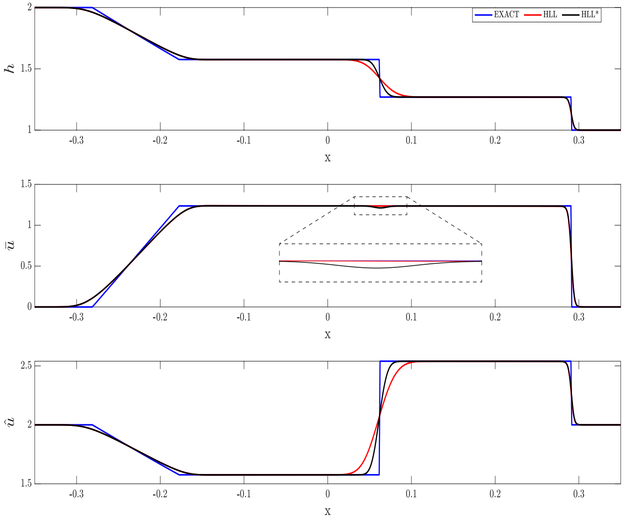
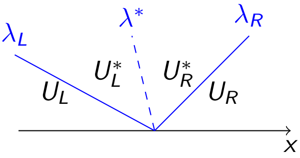


FIGURE – Dam break problem. Plots of the variables using *HLL* and *HLL** solvers for 1000 grid cells

Homogenuous 1D Shallow water model with two velocities

The 3-waves *HLLC* type scheme

$$\tilde{U}_{HLLC_{\overline{u}}}(\frac{x}{t}, U_L, U_R) = \begin{cases} U_L & \text{if } \frac{x}{t} < \lambda_L, \\ U_L^* & \text{if } \lambda_L < \frac{x}{t} < \lambda^*, \\ U_R^* & \text{if } \lambda^* < \frac{x}{t} < \lambda_R, \\ U_R & \text{if } \frac{x}{t} > \lambda_R. \end{cases}$$



The unknowns are

$$\begin{cases} U_L^* = (h_L^*, \overline{u}_L^*, \hat{u}_L^*) \\ U_R^* = (h_R^*, \overline{u}_R^*, \hat{u}_R^*) \end{cases} + \lambda^* \quad \Rightarrow \quad 7 \text{ relations.}$$

The 3-waves *HLLC* type scheme

Rankine-Hugoniot relations

- $h_L^* \overline{u}_L^* - h_L \overline{u}_L = \lambda_L \left(h_L^* - h_L \right)$ on λ_L,
- $h_R \overline{u}_R - h_R^* \overline{u}_R^* = \lambda_R \left(h_R - h_R^* \right)$ on λ_R

Riemann invariants

- $\overline{u}_L^* = \overline{u}_R^* = \lambda^*$,
- $\frac{\hat{u}_L}{h_L} = \frac{\hat{u}_L^*}{h_L^*}$ on λ_L,
- $\frac{\hat{u}_R}{h_R} = \frac{\hat{u}_R^*}{h_R^*}$ on λ_R.

The consistency relations of the discharge

$$\lambda_R h_R^* \overline{u}_R^* - \lambda_L h_L^* \overline{u}_L^* + \lambda^* \left(h_L^* \overline{u}_L^* - h_R^* \overline{u}_R^* \right) = \left(\lambda_R - \lambda_L \right) h_{HLL} \overline{u}_{HLL}.$$

Homogenuous 1D Shallow water model with two velocities

The 3-waves *HLLC* type scheme can be solved to obtain

$$\begin{cases} h_L^* = h_L(\dfrac{\lambda_L - \overline{u}_L}{\lambda_L - \overline{u}_{HLL}}), \\ h_R^* = h_R(\dfrac{\lambda_R - \overline{u}_R}{\lambda_R - \overline{u}_{HLL}}), \\ \overline{u}_L^* = \overline{u}_{HLL}, \\ \overline{u}_R^* = \overline{u}_{HLL}, \\ \hat{u}_L^* = \hat{u}_L(\dfrac{\lambda_L - \overline{u}_L}{\lambda_L - \overline{u}_{HLL}}), \\ \hat{u}_R^* = \hat{u}_R(\dfrac{\lambda_R - \overline{u}_R}{\lambda_R - \overline{u}_{HLL}}), \\ \lambda^* = \overline{u}_{HLL}. \end{cases} \tag{HLLC3}$$

For $h_L > 0$ or $h_R > 0$, we are able to prove that

- $\lambda_L < \overline{u}_{HLL} < \lambda_R$
- $h_L^* > 0$ and $h_R^* > 0$.

Numerical results

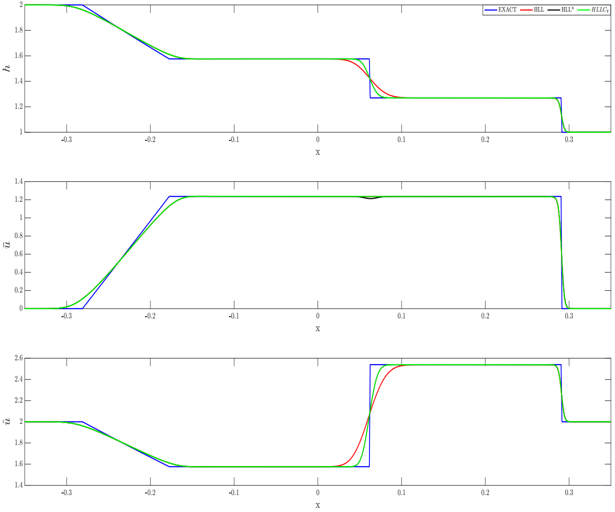


FIGURE – Dam break Problem . Plots of the variables using *HLL*, HLL^* and $HLLC_{\overline{u}}$ solvers for 1000 grid cells

Numerical results

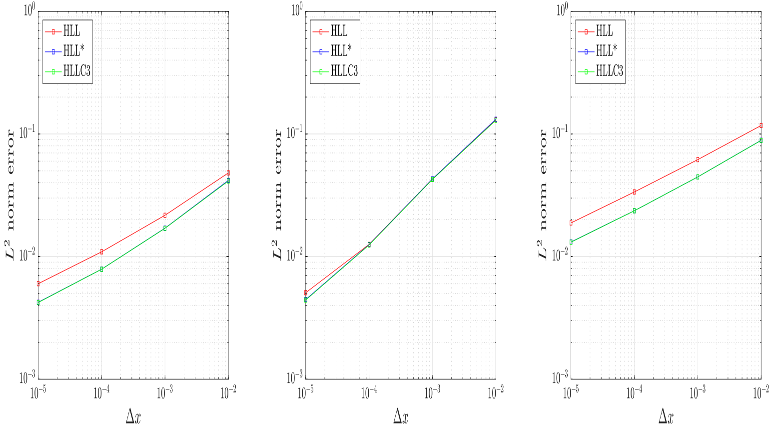


FIGURE – The Dam break problem : Convergence of L^2 norm errors at $t = 0.05s$ with mesh refinement for h on the left, for $\overline{u}$ in the middle and for $\hat{u}$ on the right.

1D Shallow water model with two velocities and topography

The model reads

$$\begin{cases} \partial_t h & + \partial_x(h\overline{u}) & = 0, \\ \partial_t(h\overline{u}) & + \partial_x(h(\overline{u}^2 + \hat{u}^2) + \frac{g}{2}h^2) & = -gh\partial_x Z, \\ \partial_t \hat{u} & + \partial_x(\overline{u}\hat{u}) & = 0. \end{cases} \qquad (SW_2)$$

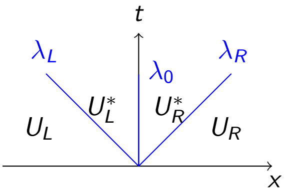
- The system has an additional eigenvalue $\lambda_0 = 0$
- The exact solution of the integral of the topography is very difficult to compute

Main Objective

To develop a "Well balanced " scheme based on the analysis of the Riemann problem

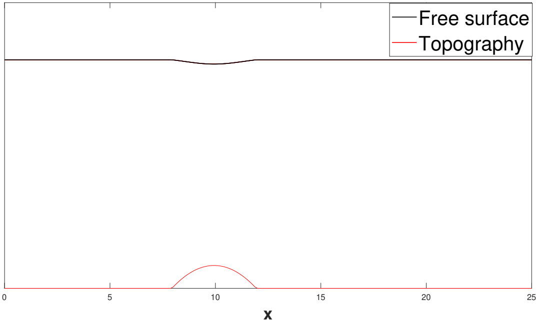
- able to excatly recover any steady solution in 1D over an arbitrary topography
- preserves the non-negativity of the water heights

Numerical schemes



- Using strategy done in (Dansac 16) and corrected in (Mbaye 21), we construct a numerical scheme named $HLLC_0$
- A new numerical scheme named $HLLC_0^*$ (in the same way of $HLL*$) is introduced

Numerical Results

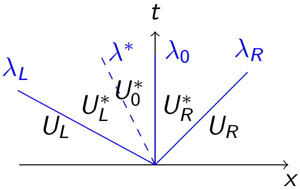


Total head and discharge $\mathcal{L}^2$ errors for the subcritical solution

	Total head	Discharge
$HLLC_0$	8.943e-11	2.46e-13
$HLLC_0^*$	8.942e-11	1.63e-13

Numerical schemes

The 4-waves *HLLC* type scheme named $HLLC_{\overline{u},0}$



- we have 3 intermediates states
- we have 10 unknowns
- we have to preserve the positivity of the water heights
- we have to assure that $\lambda_L < \lambda^* < \lambda_R$ and that λ^* keeps its sign

Conclusion and perspectives

Conclusions

- Proposed several numerical schemes for the resolution of the homogenuous (SW_2)
- Proposed briefly several numerical schemes for the resolution of the (SW_2) with the topography

Perspectives

- To construct the piecewise C^1 steady solutions of (SW_2)
- To complete the $HLLC_{\overline{u},0}$ Riemann solver
- To do the numerical scheme for the $2D$ shallow water model with two velocities
- To extend the numerical scheme for the shear shallow water model

Thankyou !