A periodic homogenization problem with defects

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Introduction

Purpose : To address an homogenization problem for a second order elliptic equation in divergence form when the coefficient is a perturbation of a periodic coefficient :

$$\begin{cases} -\operatorname{div}(a(./\varepsilon)\nabla u^{\varepsilon}) = f & \text{on } \Omega, \\ u^{\varepsilon} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Where :

- $\Omega \subset \mathbb{R}^d$ is a bounded domain ($d \geq 1$).
- $f \in L^2(\Omega)$.
- $\varepsilon > 0$ is a small scale parameter.
- a is a bounded, elliptic coefficient.

We want to identify the limit of u^{ε} when the scale parameter $\varepsilon \to 0$ and study the convergence for several topologies ($L^2(\Omega)$, $H^1(\Omega)$,...).

The periodic case

The periodic problem, when $a = a_{per}$, is well known [1] :

- u^ε converges strongly in $L^2(\Omega)$, weakly in $H^1(\Omega)$ to u^* solution to the homogenized equation :

$$\begin{cases} -\operatorname{div}(a^*_{per}\nabla u^*) = f & \text{on } \Omega, \\ u^*(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

  where (a^*_{per}) is a constant matrix.

- The behavior in $H^1(\Omega)$ is obtained introducing a corrector $w_{per,p}$ defined for all $p \in \mathbb{R}^d$ as the periodic solution (unique up to the addition of a constant) to :

$$-\operatorname{div}(a_{per}(\nabla w_{per,p} + p)) = 0 \quad \text{on } \mathbb{R}^d. \tag{3}$$

[1] [Bensoussan, Lions, Papanicolaou, 1978]

The periodic case

This corrector w_{per} allows to both make explicit the homogenized coefficient :

$$(a_{per}^*)_{i,j} = \int_Q e_i^T a_{per}(y) \left(e_j + \nabla w_{per,e_j}\right) dy,$$

and define an approximation

$$u^{\varepsilon,1} = u^* + \sum_{i=1}^{d} \varepsilon w_{per,e_i}(./\varepsilon)\partial_i u^*,$$

such that $u^{\varepsilon,1} - u^\varepsilon$ **strongly** converges to 0 in $H^1(\Omega)$.

Essential property of w_{per} : **strict sub-linearity at infinity**,

$$\varepsilon w_{per}(./\varepsilon) \to 0, \quad \text{when } \varepsilon \to 0.$$

The perturbed problem

- Our purpose is to extend these results to the problem when $a \neq a_{per}$ describes a perturbed periodic background.
- Typical class of perturbed coefficients :

$$a = a_{per} + \tilde{a}.$$

- **Main difficulty** : The corrector equation

$$-\operatorname{div}\left(a\left(\nabla w_p + p\right)\right) = 0 \text{ in } \mathbb{R}^d,$$

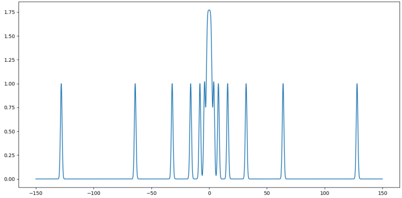
cannot be reduced to an equation posed on a bounded domain as is the case in periodic context, which prevents us from using classical techniques (Poincaré Inequality, Lax-Milgram Lemma...).

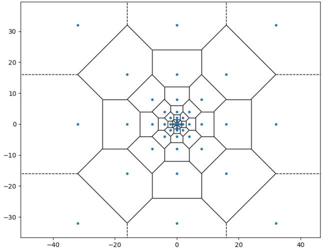
The perturbed problem

Extensions in the case $a = a_{per} + \tilde{a}$:

First extension [2]: Case of local perturbations (i.e $\tilde{a}(x) \to 0$ when $|x| \to \infty$) when $\tilde{a} \in L^r(\mathbb{R}^d)$ for $r \in]1, \infty[$.

Second extension [3] : Case of non-local perturbations that become rare at infinity.





[2] [Blanc, Le Bris, Lions, 2012, 2018] & [Blanc, Josien, Le Bris, 2020]
[3] [Goudey, 2020]

The perturbed problem

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Second extension [3] : Case of non-local perturbations that become rare at infinity.

- The homogenized limit is identical to that of the periodic case without defect ($\tilde{a} = 0$). The existence of an adapted corrector w_p is established.
- $w_p = w_{per,p} + \tilde{w}_p$, where $w_{per,p}$ is the periodic corrector and ∇w_p shares the same structure as the coefficient a (ie. $\nabla \tilde{w}_p \in L^r(\mathbb{R}^d)$ or $\nabla \tilde{w}_p$ becomes rare at infinity).

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The case $\delta a \in L^r(\mathbb{R}^d)$

Denote $Q = (]0,1[)^d$. We consider here a perturbed periodic background described by :

$$\delta a := (a(. + e_i) - a)_{i \in \{1,\ldots,d\}} \in \left(L^r(\mathbb{R}^d)\right)^d, \quad r \in [1, +\infty[. \tag{A1}$$

- Motivation : δ measures the deviation from a periodic background $\rightarrow$ Already used in the literature to study some properties of solutions to periodic elliptic equations [4].
- Here (A1) ensures that the coefficient behaves as a Q-periodic function at infinity.

Questions : Limit of u^ε when $\varepsilon \to 0$? Existence of an adapted corrector ?

[4] In particular in [Moser, Struwe, 1992]

We first consider that $\nabla a \in \left(L^r(\mathbb{R}^d)\right)^d$.

- Assume $r < d$, and denote $r^* = \dfrac{rd}{d-r}$.
  Gagliardo-Nirenberg-Sobolev inequality : $\exists c \in \mathbb{R}, M > 0$ s.t.

$$\|a - c\|_{L^{r^*}(\mathbb{R}^d)} \leq M \|\nabla a\|_{\left(L^r(\mathbb{R}^d)\right)^d}.$$

  Then $\quad a = c + a - c = a_{per} + \tilde{a} \quad$ with $\tilde{a} \in L^{r^*}(\mathbb{R}^d)$.
  $\Rightarrow$ Particular case of the setting of local defects in $L^{r^*}(\mathbb{R}^d)$!

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- Assume $r \geq d$: Existence of coefficients a such that u^ε only converges along a sub-sequence (and can have several adherent values) :

$$\text{Examples : } \begin{aligned} a(x) &= 2 + \sin(\ln(1+|x|)) && \text{for } d = 1, r > 1, \\ a(x) &= 2 + \sin(\ln(\ln(2+|x|))) && \text{for } d = 2, r = 2. \end{aligned}$$

The case $\delta a \in L^r(\mathbb{R}^d)$, $r < d$

For $f \in L^1_{loc}(\mathbb{R}^d)$, we denote $\mathcal{M}(f)(x) := \int_{Q+x} f(y)dy$.

Lemma 1 : Discrete Gagliardo-Nirenberg-Sobolev inequality

Assume $r < d$. Then, for every f such that $\delta f \in \left(L^r(\mathbb{R}^d)\right)^d$, there exists a Q-periodic function f_{per} and a constant $C > 0$ independent of f such that

$$\|\mathcal{M}(|f - f_{per}|)\|_{L^{r^*}(\mathbb{R}^d)} \leq C\|\delta f\|_{\left(L^r(\mathbb{R}^d)\right)^d}.$$

- Proof : Adaptation of the continuous GNS inequality in our discrete setting.
- Ensures a control of the behavior of local averages of f at infinity. $\rightarrow$ Up to a local averaging, the function $f - f_{per}$ is an L^{r^*} function at infinity.

The case $\delta a \in L^r(\mathbb{R}^d)$, $r < d$

For $r < d$, we define the following space of perturbations :

$$\mathcal{A}^r(\mathbb{R}^d) = \left\{ f \in L^1_{loc}(\mathbb{R}^d) \;\middle|\; \mathcal{M}(|f|) \in L^{r^*}(\mathbb{R}^d) \text{ and } \delta f \in \left(L^r(\mathbb{R}^d)\right)^d \right\},$$

equipped with the norm :

$$\|f\|_{\mathcal{A}^r(\mathbb{R}^d)} = \|\mathcal{M}(|f|)\|_{L^{r^*}(\mathbb{R}^d)} + \|\delta f\|_{\left(L^r(\mathbb{R}^d)\right)^d}.$$

Lemma 1 $\Rightarrow$ If $\delta a \in \left(L^r(\mathbb{R}^d)\right)^d$, there exists a_{per} s.t. :

$$a = a_{per} + a - a_{per} = a_{per} + \tilde{a} \quad \text{with } \tilde{a} \in \mathcal{A}^r(\mathbb{R}^d).$$

The case $\delta a \in L^r(\mathbb{R}^d)$, $r < d$

- For $r < d$ we study the perturbed problem when $a = a_{per} + \tilde{a}$ with $\tilde{a} \in \mathcal{A}^r(\mathbb{R}^d)$.
- Motivation : Existence of coefficients $\tilde{a} \in \mathcal{A}^r$ such that $\tilde{a} \notin L^{r^*}(\mathbb{R}^d)$.

Proposition (Average)

Let $f \in \mathcal{A}^r(\mathbb{R}^d)$, then $\langle |f| \rangle = \lim_{R\to\infty} \dfrac{1}{|\mathrm{B}_R|} \displaystyle\int_{\mathrm{B}_R} |f(x)|dx = 0$.

- On average, a perturbation belonging to $\mathcal{A}^r(\mathbb{R}^d)$ does not affect the periodic background. The homogenized limit of u^ε is therefore expected to be the same as in the case $a = a_{per}$.

The case $\delta a \in L^r(\mathbb{R}^d)$, $r < d$

Theorem 1 : Existence result for the corrector equation

Assume $r < d$, $a_{per} \in L^2_{per}(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$, $\tilde{a} \in \mathcal{A}^r(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$, $\alpha \in]0,1[$.
Then, for every $p \in \mathbb{R}^d$, there exists a unique (up to an additive constant) function $w_p \in L^1_{loc}(\mathbb{R}^d)$ solution to :

$$\begin{cases} -\operatorname{div}((a_{per} + \tilde{a})(p + \nabla w_p)) = 0 \quad \text{on } \mathbb{R}^d, \\ \lim\limits_{|x| \to \infty} \dfrac{|w_p(x)|}{1 + |x|} = 0, \end{cases} \tag{4}$$

such that $\nabla w_p \in \left(L^2_{per}(\mathbb{R}^d) + \mathcal{A}^r(\mathbb{R}^d)\right)^d \cap \left(\mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$.

Main Idea : Assume $w_p = w_{p,per} + \tilde{w}_p$ with $\nabla \tilde{w}_p \in (\mathcal{A}^r(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d))^d$, then (4) is equivalent to an equation of the form :

$$-\operatorname{div}((a_{per} + \tilde{a})\nabla u) = \operatorname{div}(f) \quad \text{in } \mathbb{R}^d,$$

where $u = \tilde{w}_p$ and $f = \tilde{a}\,(p + \nabla w_{p,per}) \in (\mathcal{A}^r(\mathbb{R}^d))^d$.

The case $\delta a \in L^r(\mathbb{R}^d)$, $r < d$

Key Lemma for the proof [5] of Theorem 1 :

Lemma 2 : A priori estimate

With the assumptions of Theorem 1, there exists $C > 0$ such that for every $f \in (\mathcal{A}^r(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d))^d$ and u solution in $\mathcal{D}'(\mathbb{R}^d)$ to

$$-\operatorname{div}((a_{per} + \tilde{a})\nabla u) = \operatorname{div}(f) \quad \text{in } \mathbb{R}^d,$$

with $\nabla u \in \left(\mathcal{A}^r(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^d$, we have the following estimate :

$$\|\delta \nabla u\|_{\left(L^r(\mathbb{R}^d)\right)^{d\times d}} + \|\nabla u\|_{\left(C^{0,\alpha}(\mathbb{R}^d)\right)^d} \leq C\left(\|\delta f\|_{\left(L^r(\mathbb{R}^d)\right)^{d\times d}} + \|f\|_{\left(C^{0,\alpha}(\mathbb{R}^d)\right)^d}\right)$$

- Lemma 2 ensures the continuity of the reciprocal linear operator $f \mapsto \nabla\left(-\operatorname{div} a\nabla\right)^{-1}\operatorname{div}(f)$ from $\mathcal{A}^r(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$ to $\mathcal{A}^r(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)$.

[5] Proof adapted from [Blanc, Le Bris, Lions, 2018]

The case $\delta a \in L^r(\mathbb{R}^d)$, $r < d$

Theorem 2 : Homogenization results

Let $r < d$. Assume $\delta a \in \left(L^r(\mathbb{R}^d)\right)^d$ and regularity properties. Let u^ε the sequence of solutions in $H_0^1(\Omega)$ to

$$\left\{ \begin{array}{cc} -\operatorname{div}(a(./\varepsilon)\nabla u^\varepsilon) = f & \text{on } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \end{array} \right.$$

Then, there exists a Q-periodic function a_{per} such that u^ε converges (weak-$H^1(\Omega)$ and strong-$L^2(\Omega)$) to u^* solution to

$$\left\{ \begin{array}{cc} -\operatorname{div}(a^*_{per}\nabla u^*) = f & \text{on } \Omega \\ u^*(x) = 0. & \text{on } \partial\Omega. \end{array} \right.$$

The case $\delta a \in L^r(\mathbb{R}^d)$, $r < d$

Theorem 2 : Convergence results

Assume $r < d$, $r^* \neq d$ and Ω is a $C^{1,1}$-bounded domain. Let $\Omega_1 \subset\subset \Omega$.
We define $u^{\varepsilon,1} = u^* + \varepsilon \displaystyle\sum_{i=1}^{d} \partial_i u^* w_{e_i}(./\varepsilon)$ where w_{e_i} is solution to corrector equation for $p = e_i$ and $u^* \in H^1(\Omega)$ is the homogenized limit. Then $R^\varepsilon = u^\varepsilon - u^{\varepsilon,1}$ satisfies the following estimates :

$$\|R^\varepsilon\|_{L^2(\Omega)} \leq C_1 \varepsilon^{\nu_r} \|f\|_{L^2(\Omega)},$$

$$\|\nabla R^\varepsilon\|_{L^2(\Omega_1)} \leq C_2 \varepsilon^{\nu_r} \|f\|_{L^2(\Omega)},$$

$$\nu_r = \min\left(1, \frac{d}{r^*}\right) \in]0,1],$$

where C_1 and C_2 are two positive constants independent of f and ε.

A counter example for $r \geq d$

- If $r \geq d$: Existence of coefficients a such that $\delta a \in \left(L^r(\mathbb{R}^d)\right)^d$ and u^ε has at least two adherent values.
- In this case $a \neq a_{per} + \tilde{a}$ where $\tilde{a}$ is "integrable".
- Example for $d = 1$, $r > 1$: $a(x) = 2 + \sin(\ln(1 + |x|))$, $\Omega =]1, 2[$

1. $\delta a(x) = O\left(\frac{1}{1+|x|}\right) \Rightarrow \delta a \in L^r(\mathbb{R}),\ \forall r > 1.$
2. If $\varepsilon_n = e^{-2n\pi}$, $u^{\varepsilon_n} \overset{n\to\infty}{\longrightarrow} u_1^*$ solution to $-\frac{d}{dx}\left(a_1^* \frac{d}{dx} u_1^*\right) = f$, where $a_1^*(x) = 2 + \sin(\ln(x))$.
3. If $\varepsilon_n = e^{-(2n+1)\pi}$, $u^{\varepsilon_n} \overset{n\to\infty}{\longrightarrow} u_2^*$ solution to $-\frac{d}{dx}\left(a_2^* \frac{d}{dx} u_2^*\right) = f$, where $a_2^*(x) = 2 - \sin(\ln(x))$.

Thank you for your attention !