Symmetry reduction in AM/GM-based optimization

Philippe Moustrou, UiT - The Arctic University of Norway
Joint work with H. Naumann, C. Riener, T. Theobald and H. Verdure

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$\rightarrow$ Symmetry reduction for certificates based on SAGE functions.

[M., Naumann, Riener, Theobald, Verdure, 2021^+]

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→ What about other certificates?

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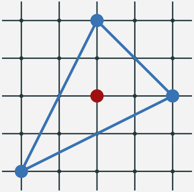

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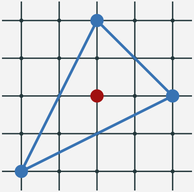

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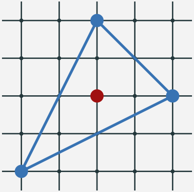

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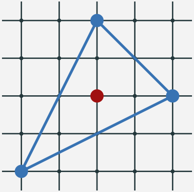

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→ The more general framework of signomials.

→ An AGE signomial is a sum of exponentials of the form

$$f(x) = \sum_{\alpha \in \mathcal{A}} c_\alpha e^{\langle \alpha, x \rangle} + c_\beta e^{\langle \beta, x \rangle}$$

such that $\mathcal{A} \cup \{\beta\} \subset \mathbb{R}^n$, $c_\alpha \geq 0$, $c_\beta \in \mathbb{R}$, and $f(x) \geq 0$ on $\mathbb{R}^n$.

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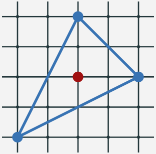

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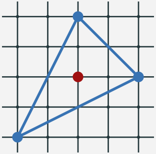

- $D(\nu, e \cdot c) \leqslant c_\beta$,

where $D(\nu, e \cdot c) = \sum_{\alpha \in \mathcal{A}} \nu_\alpha \ln \left(\frac{\nu_\alpha}{e \cdot c_\alpha} \right)$ is the relative entropy function.

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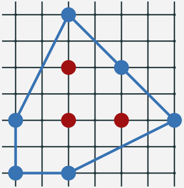
→ Can be solved with relative entropy programming.

Size of the problem



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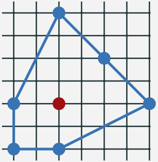

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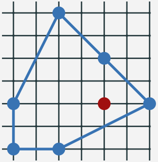

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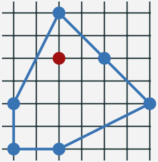

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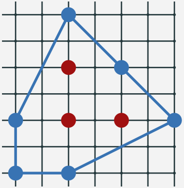

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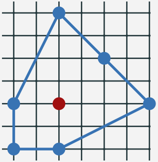

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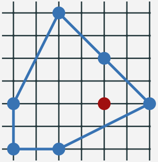

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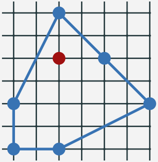

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$\rightarrow 2|\mathcal{B}||\mathcal{A}|$ variables.

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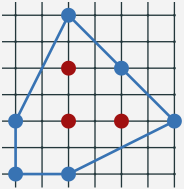

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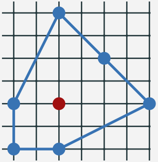

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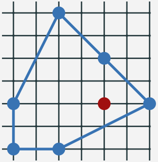

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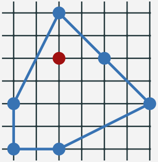

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$\rightarrow$ $2|\mathcal{B}||\mathcal{A}|$ variables.

$\rightarrow$ $n|\mathcal{B}| + |\mathcal{B}| + |\mathcal{A}|$ constraints.

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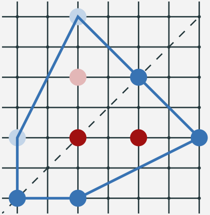
→ Let $\hat{\mathcal{A}}$, $\hat{\mathcal{B}}$ be sets of G-representatives.



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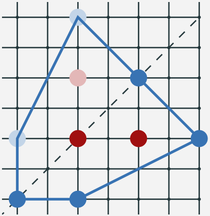

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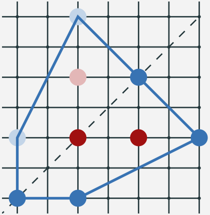

→ Does f have a symmetric decomposition?

→ Can we reduce the size of the relative entropy program?

Orbit decomposition

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Theorem [M., Naumann, Riener, Theobald, Verdure]

The signomial f is a SAGE if and only if for every $\hat{\beta} \in \hat{\mathcal{B}}$, there exists an AGE signomial $h_{\hat{\beta}}$ such that

$$f = \sum_{\hat{\beta} \in \hat{\mathcal{B}}} \sum_{\rho \in G / \operatorname{Stab}(\hat{\beta})} \rho h_{\hat{\beta}}.$$

The functions $h_{\hat{\beta}}$ can be chosen invariant under the action of $\operatorname{Stab}(\hat{\beta})$.

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$\rightarrow$ Moreover, the invariance under $\operatorname{Stab}(\hat{\beta})$ allows to further reduce the number of variables and constraints.

Symmetry reduction

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Theorem [M., Naumann, Riener, Theobald, Verdure]

The signomial f is a SAGE if and only if for every $\hat{\beta} \in \hat{\mathcal{B}}$, there exist $c^{(\hat{\beta})} \in \mathbb{R}_+^{\mathcal{A}/\operatorname{Stab}(\hat{\beta})}$ and $\nu^{(\hat{\beta})} \in \mathbb{R}_+^{\mathcal{A}/\operatorname{Stab}(\hat{\beta})}$ such that

(i) $\sum_{\alpha \in \mathcal{A}/\operatorname{Stab}(\hat{\beta})} \nu_\alpha^{(\hat{\beta})} \sum_{\alpha' \in \operatorname{Stab}(\hat{\beta}) \cdot \alpha} (\alpha' - \hat{\beta}) = 0 \qquad \forall\ \hat{\beta} \in \hat{\mathcal{B}},$

(ii) $\sum_{\alpha \in \mathcal{A}/\operatorname{Stab}(\hat{\beta})} \left| \operatorname{Stab}(\hat{\beta}) \cdot \alpha \right| \nu_\alpha^{(\hat{\beta})} \ln \frac{\nu_\alpha^{(\hat{\beta})}}{e c_\alpha^{(\hat{\beta})}} \leqslant c_{\hat{\beta}} \qquad \forall\ \hat{\beta} \in \hat{\mathcal{B}},$

(iii) $\sum_{\hat{\beta} \in \hat{\mathcal{B}}} \frac{|\operatorname{Stab}(\alpha)|}{|\operatorname{Stab}(\hat{\beta})|} \sum_{\gamma \in (G \cdot \alpha)/\operatorname{Stab}(\hat{\beta})} \left| \operatorname{Stab}(\hat{\beta}) \cdot \gamma \right| c_\gamma^{(\hat{\beta})} \leqslant c_\alpha \qquad \forall\ \alpha \in \hat{\mathcal{A}}.$

Size estimate

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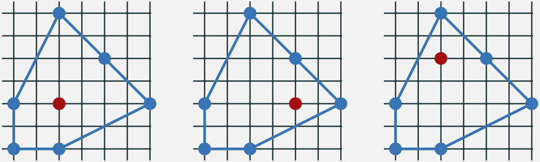
$\rightarrow$ Without reduction: $2|\mathcal{B}||\mathcal{A}|$ variables, $n|\mathcal{B}| + |\mathcal{B}| + |\mathcal{A}|$ constraints.



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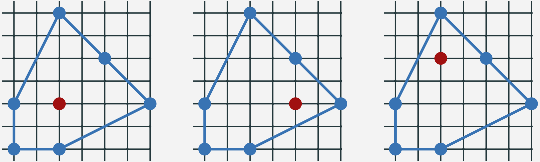

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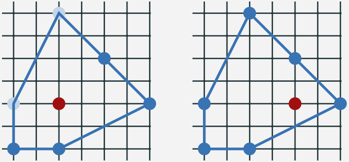

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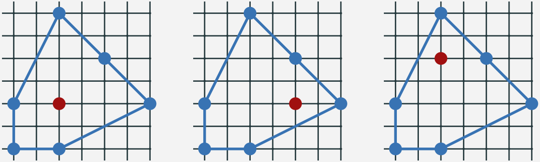

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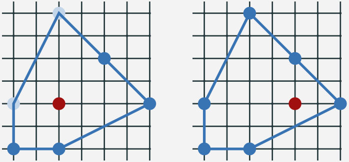

→ $2\sum_{\hat{\beta}\in\hat{\mathcal{B}}} |\mathcal{A}/\operatorname{Stab}(\hat{\beta})|$ variables.

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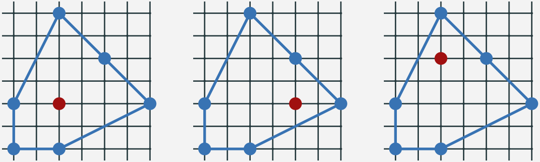

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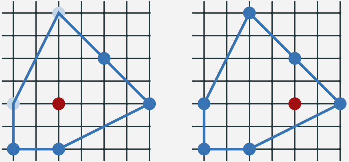

→ $2\sum_{\hat{\beta}\in\hat{\mathcal{B}}}|\mathcal{A}/\operatorname{Stab}(\hat{\beta})|$ variables.

→ At most $n|\hat{\mathcal{B}}| + |\hat{\mathcal{B}}| + |\hat{\mathcal{A}}|$ constraints.

A stability result

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→ For $\alpha \in \mathbb{R}^n$, denote by $\mathrm{wt}(\alpha)$ its number of non-zero coordinates.

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$\rightarrow$ For $\alpha \in \mathbb{R}^n$, denote by $\mathrm{wt}(\alpha)$ its number of non-zero coordinates.

Theorem [M., Naumann, Riener, Theobald, Verdure]

Let $k, \ell, w \in \mathbb{N}$ be fixed. Then for every integer $n \geqslant 2w$ and every $\mathcal{S}_n$-invariant signomial such that $|\hat{\mathcal{A}}| \leqslant k$, $|\hat{\mathcal{B}}| \leqslant \ell$, and

$$\max_{\hat{\gamma} \in \hat{\mathcal{A}} \cup \hat{\mathcal{B}}} \mathrm{wt}(\hat{\gamma}) \leqslant w,$$

the number of constraints and the number of variables of the symmetry adapted program are bounded by constants only depending of k, ℓ and w:

$$C_n \leqslant k + \ell + \ell(w+1) \quad \text{and} \quad V_n \leqslant 2\ell k u(w),$$

where $u(w) = \sum_{i=0}^{w} \binom{w}{i}^2 i!\,.$

Concrete size comparisons

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$\rightarrow$ Look at some cases with $\hat{\mathcal{A}} = \{0, \hat{\alpha}\}$ and $\hat{\mathcal{B}} = \{\hat{\beta}\}$

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		Standard		Symmetric	
$\lvert\mathcal{S}_n \cdot \hat{\beta}\rvert$	$\lvert\mathcal{S}_n \cdot \hat{\alpha}\rvert$	V_n	C_n	V_n	C_n
1	$n!$	$2n! + 3$	$n! + n + 2$	5	4
$n!$	n	$2(n+1)n! + 1$	$(n+1)(n!+1)$	$2n+3$	$n+3$
$n!$	$n!$	$2(n!+1)n! + 1$	$n!(n+2) + 1$	$2n! + 3$	$n+3$
n	n	$2n(n+1) + 1$	$(n+1)^2$	7	5

A numerical example

A numerical example

$\rightarrow$ $\hat{\alpha} = (n^2, 0, \ldots, 0)$, $\hat{\beta} = (1, 2, \ldots, n)$.

		Standard method				**Symmetric method**			
dim	bound	V_n	C_n	t_s	t_r	V_n	C_n	t_s	t_r
2	-0.2109	13	9	0.0173	0.0185	7	5	0.0297	0.0311
3	-0.8888	49	28	0.0427	0.0454	9	6	0.0248	0.0264
4	-4.111	241	125	0.152	0.1701	11	7	0.0296	0.0318
5	-22.30	1441	726	0.7888	0.8433	13	8	0.0356	0.0384
6	-141.0	10081	5047	5.422	5.843	15	9	0.0423	0.0458
7	-1024	80641	40328	57.26	66.67	17	10	0.0491	0.0538
8	-8418	725761	362889	1514	2211	19	11	0.0568	0.0626
9	-77355	7257601	3628810	–	–	21	12	0.0661	0.0835
10		79833601	39916811	–	–	23	13	–	–

Thank you!