Mountain pass 1D connections for Allen-Cahn systems

Oliver-Bonafoux, Ramon.
ramon.oliver_bonafoux@sorbonne_universite.fr

LJLL, Sorbonne Université

June 22, 2021

PhD advisor: Fabrice Bethuel (LJLL, Sorbonne Université).

This program has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 754362.

Allen-Cahn systems.

$$(\partial_t u_\varepsilon) - \Delta u_\varepsilon = -\varepsilon^{-2}\nabla_u V(u_\varepsilon), \text{ in } ([0,T]) \times \Omega \tag{1}$$

where

- $u_\varepsilon : ([0,T]) \times \Omega \to \mathbb{R}^k$. We are interested in $k \geq 2$.
- $\Omega \subset \mathbb{R}^N$.
- $\varepsilon > 0$.
- $V : \mathbb{R}^k \to \mathbb{R}$ a non-negative, smooth **multi-well potential**.

Interest: Phase transitions (at $\varepsilon \approx 0$), reaction-diffusion systems.

Standard assumptions on V:

- $V \in \mathcal{C}^2(\mathbb{R}^k)$, $V \geq 0$. $\Sigma := \{V = 0\} = \{\sigma_1, \ldots, \sigma_l\}$, $2 \leq l < +\infty$.
- V is not 0 at ∞.
- The **wells** (elements of Σ) are non-degenerate.

Of course, V is **non-convex**!

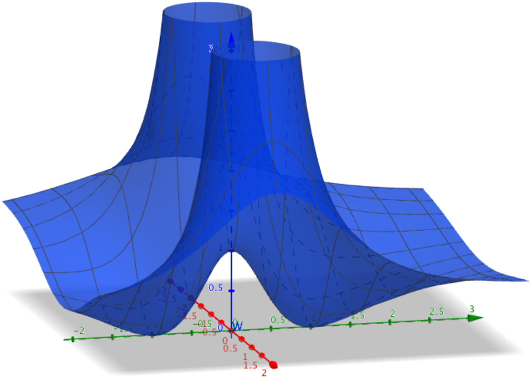

Figure: **Double-well potential**.

Problem: **ODE system**:

$$\mathfrak{q}'' = \nabla_u V(\mathfrak{q}), \text{ in } \mathbb{R}. \tag{2}$$

Associated **energy functional**:

$$E(q) := \int_{\mathbb{R}} \left(\frac{|q'(t)|^2}{2} + V(q(t)) \right) dt, \quad q \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^k).$$

Connection: a solution of (2), $\mathfrak{q}$, such that $E(\mathfrak{q}) < +\infty$. Moreover

$$\lim_{t \to \pm\infty} \mathfrak{q}(t) \in \Sigma.$$

Two possibilities:

- $\mathfrak{q}(-\infty) \neq \mathfrak{q}(+\infty)$. **Heteroclinic** case.
- $\mathfrak{q}(-\infty) = \mathfrak{q}(+\infty)$. **Homoclinic** case.

Our goal: Existence of non-minimizing connections.

Features of the energy:

$$E(q) := \int_{\mathbb{R}} \left(\frac{|q'(t)|^2}{2} + V(q(t)) \right) dt, \quad q \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^k).$$

Assumptions on V imply:

- **Non-negativity**: $E \geq 0$ on $H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^k)$.
- **Invariance by translations**: For all $q \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^k)$ and $\tau \in \mathbb{R}$, we have

$$E(q(\cdot + \tau)) = E(q).$$

- **Finite energy functions connect Σ at $\pm\infty$**: If $q \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^k)$ is such that $E(q) < +\infty$, then

$$\lim_{t \to \pm\infty} q(t) \in \Sigma.$$

Natural functional spaces for E:

$$X(\sigma_i,\sigma_j) := \{q \in H^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^k) : E(q) < +\infty \\ \lim_{t\to-\infty} q(t) = \sigma_i \text{ and } \lim_{t\to+\infty} q(t) = \sigma_j\},$$

for $(\sigma_i,\sigma_j) \in \Sigma^2$. The properties of the energy imply:

$$\{q \in H^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^k) : E(q) < +\infty\} = \bigcup_{(\sigma_i,\sigma_j)\in\Sigma^2} X(\sigma_i,\sigma_j)$$

and that for any $(\sigma_i,\sigma_j) \in \Sigma^2$, $\psi_{(\sigma_i,\sigma_j)} \in X(\sigma_i,\sigma_j)$

$$X(\sigma_i,\sigma_j) = \{\psi_{(\sigma_i,\sigma_j)}\} + H^1(\mathbb{R},\mathbb{R}^k).$$

Recall also the **continuous embedding**

$$i : H^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^k) \to \mathcal{C}^{0,\frac{1}{2}}_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^k).$$

First idea: Minimization. Study the problem

$$\mathfrak{m}_{\sigma_i\sigma_j} := \inf_{q \in X(\sigma_i,\sigma_j)} E(q)$$

for $(\sigma_i, \sigma_j) \in \Sigma^2$. Two diferent **cases**:

1. $\sigma_i = \sigma_j = \sigma$. Straightforward: $\mathfrak{m}_{\sigma_i\sigma_j} = 0$, attained by the **constant homoclinic** σ.
2. $\sigma_i \neq \sigma_j$. $\mathfrak{m}_{\sigma_i\sigma_j}$ is **attained** in the double-well case. Otherwise, one imposes (because of the lack of compactness)

$$\mathfrak{m}_{\sigma_i\sigma_j} < \min\{\mathfrak{m}_{\sigma_i\sigma} + \mathfrak{m}_{\sigma\sigma_j} : \sigma \in \Sigma \setminus \{\sigma_i, \sigma_j\}\}.$$

References: Schatzman (2002), Alama, Bronsard, Contreras and Pelinovsky (2015), Monteil and Santambrogio (2018), Zuñiga and Sternberg (2016).

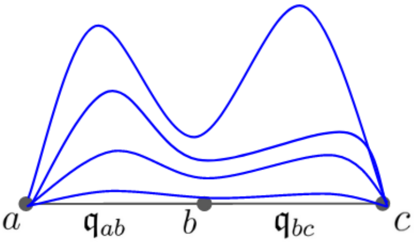

Figure: If there is no strict triangle's inequality, minimizing sequences may always tend to a gluing of heteroclinics involving other wells.

Seeking for non-minimizing connections: **From now on**, fix $(\sigma^-, \sigma^+) \in \Sigma^2$, $\sigma^- \neq \sigma^+$. Assume the strict triangle's inequality.

- If $k = 1$, heteroclinics are at most unique (up to translations) and global minimizers of E. **Non-minimizing** connections **cannot** exist.
- For $k \geq 2$, several globally minimizing heteroclinics can exist. Striking consequences: Alama, Bronsard and Gui (1997) (ABG), a counterexample of **De Giorgi's conjecture** in the vector-valued case.

Conclusion: We restrict to $k \geq 2$, and assume **multiplicity** of globally minimizing heteroclinics as follows

$$\mathcal{F} := \{\mathfrak{q} \in X(\sigma^-, \sigma^+) : E(\mathfrak{q}) = \mathfrak{m}_{\sigma^-\sigma^+}\} = \mathcal{F}_0 \cup \mathcal{F}_1$$

with $\mathcal{F}_0$ and $\mathcal{F}_1$ **non-empty** such that $\mathrm{dist}_{H^1(\mathbb{R},\mathbb{R}^k)}(\mathcal{F}_0, \mathcal{F}_1) > 0$.

First observation: **Multiplicity** of minimizing heteroclinics and **compactness** of minimizing sequences imply the existence of **a Mountain pass geometry**:

Proposition (R. O.-B. (2021))

Under general assumptions (including strict triangle's inequality), it holds

$$\mathfrak{c} := \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} J(\gamma(s)) > \mathfrak{m}_{\sigma^- \sigma^+},$$

where

$$\Gamma := \{\gamma \in C^0([0,1], H^1(\mathbb{R}, \mathbb{R}^k)) : \gamma(i) \in \mathcal{F}_i \text{ for } i \in \{1, 2\}\}.$$

The **functional**

$$J : v \in H^1(\mathbb{R}, \mathbb{R}^k) \to E(\psi + v) \in \mathbb{R}^+$$

is C^1, where $\psi \in X(\sigma^-, \sigma^+)$

Mountain Pass Lemma: there exists a **Palais-Smale sequence** $(q_n)_{n\in\mathbb{N}}$ in $X(\sigma^-,\sigma^+)$ such that

$$J(q_n-\psi)\to \mathfrak{c} \text{ and } DJ(q_n-\psi)\to 0 \text{ in } H^1(\mathbb{R},\mathbb{R}^k)$$

(identifying via Riesz's Theorem).

Question: Does J satisfy the **Palais-Smale condition** at $\mathfrak{c}$? Not in general, even up to translations $\Rightarrow$ We need to understand how the **Palais-Smale sequences** behave.

Using standard results: For any $(\tau_n)_{n\in\mathbb{N}}$ there exists $\mathfrak{q} \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^k)$ solving

$$\mathfrak{q}'' = \nabla_u V(\mathfrak{q}) \text{ in } \mathbb{R}$$

such that (up to subsequences) $q_n(\cdot + \tau_n) - \mathfrak{q} \to 0$ strongly in $H^1(S_K, \mathbb{R}^k)$ for any $S_K \subset \mathbb{R}$ **compact** and $E(\mathfrak{q}) \leq \mathfrak{c}$. It could be $\mathfrak{q}$ constant or $\mathfrak{q} \in \mathcal{F}$. **We need to rule out these possibilities**.

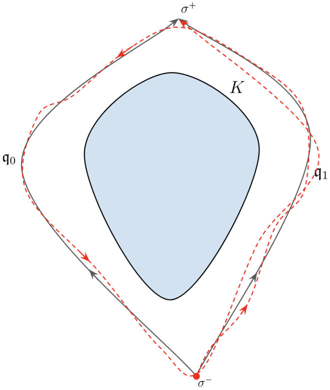

Figure: We want to exclude this behavior for the Palais-Smale sequence, where $\mathfrak{q}_0$ and $\mathfrak{q}_1$ are two minimizing heteroclinics.

We introduce additional assumptions. Either:

1. $\mathfrak{c} \notin \{(2k+1)\mathfrak{m}_{\sigma^-\sigma^+} : k \in \mathbb{N}^*\}$. A more **general** (and technical) condition works, using a **refinement** of the Mountain Pass Lemma due to Ghoussoub and Preiss (1989).
2. Using the assumptions in ABG: V **symmetric** with respect to a reflection.

Then, a **non-minimizing solution** $\mathfrak{u}$ exists. At least one of the following two possibilities holds:

- Exists $\mathfrak{u} \in X(\sigma^-, \sigma^+)$ a solution such that $E(\mathfrak{u}) > \mathfrak{m}_{\sigma^-\sigma^+}$.
- Exists $\mathfrak{u} \in X(\sigma, \sigma)$, $\sigma \in \{\sigma^-, \sigma^+\}$, **non-constant homoclinic solution**.

Remarks:

- In any case, we need

$$\mathfrak{c} < \min\{\mathfrak{m}_{\sigma_i \sigma} + \mathfrak{m}_{\sigma \sigma_j} : \sigma \in \Sigma \setminus \{\sigma_i, \sigma_j\}\}$$

  which is void for **double-well potentials**.
- Combining 1. and 2., there exists **a non-minimizing heteroclinic solution**.

Why do the assumptions work? The key ideas:

1. If the Palais-Smale sequence tends to a **concatenate** of globally minimizing heteroclinics, then $\mathfrak{c} \in \{(2k+1)\mathfrak{m}_{\sigma^-\sigma^+} : k \in \mathbb{N}^*\}$.
2. If V is symmetric with respect to the reflexion exchanging σ^- and σ^+, we can restrict to a smaller subspace of equivariant functions.

Open questions:

- Are the additional assumptions that really necessary?
- It is possible to say more about the **behavior** of the solutions? (For instance, rule out completely the homoclinic case).
- Consider $V : \mathcal{M} \to \mathbb{R}$, $\mathcal{M}$ a finite-dimensional manifold. Does some **complexity of the topology of** $\mathcal{M}$ imply the existence of **non-minimizing connections**?

Thank you for your attention!