Extension of injective immersions with application to numerical inversion and observer design

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Motivation 1 : numerical resolution of equations

Given $T : \mathcal{S} \subset \mathbb{R}^n \to T(\mathcal{S}) \subset \mathbb{R}^m$ an **injective immersion** such that $0 \in T(\mathcal{S})$

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Minimize $J(x) = T(x)^\top T(x)$ on $\mathcal{S}$

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Otherwise : local minima or solutions tending to $\partial\mathcal{S}$ possible

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$\Rightarrow$ works if T diffeomorphism $(m = n)$ and $T(\mathcal{S}) = \mathbb{R}^n$

Motivation 2 : observer design

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Whatever u in $\mathcal{U}$, whatever the initial condition $x_0 \in \mathcal{X}_0$, find an estimate $\hat{x}(t)$ of $x(t)$ at each time $t \geq 0$, based on $u_{[0,t]}$ and $y_{[0,t]}$ and such that

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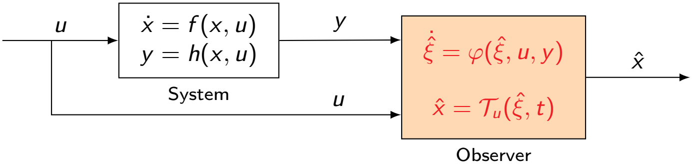

x-coordinates $\qquad n \leq m \qquad$ ξ-coordinates

$$\begin{array}{rcl} \dot{x} & = & f(x,u) \\ y & = & h(x,u) \end{array} \xrightarrow{\quad T \quad} \begin{array}{rcl} \dot{\xi} & = & F(\xi,u,y) \\ y & = & H(\xi,u) \end{array}$$

$x \in \mathcal{S} \subset \mathbb{R}^n \qquad \xi \in T(\mathcal{S}) \subset \mathbb{R}^m$

$$\hat{x} = \ \textbf{?} \xleftarrow{\quad \mathcal{T}\ \textbf{?} \quad} \dot{\hat{\xi}} = \varphi(\hat{\xi}, u, y)$$

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Can we write $\dot{\hat{\xi}}$ directly in the x-coordinates to avoid the inversion problem ?

General idea : augmentation into surjective diffeomorphism

$\boldsymbol{T} : \boldsymbol{\mathcal{S}} \subset \mathbb{R}^{\boldsymbol{n}} \to \boldsymbol{T}(\boldsymbol{\mathcal{S}}) \subset \mathbb{R}^{\boldsymbol{m}}$ **injective immersion**

Idea : add $m - n$ dimensions to the state x through a new variable $\boldsymbol{w} \in \mathbb{R}^{\boldsymbol{m-n}}$ and augment T into a diffeomorphism $\boldsymbol{T_e} : \boldsymbol{\mathcal{S}} \times \boldsymbol{\mathcal{S}_w} \to \boldsymbol{T_e}(\boldsymbol{\mathcal{S}} \times \boldsymbol{\mathcal{S}_w}) \subset \mathbb{R}^{\boldsymbol{m}}$ such that

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=> OK if $\boldsymbol{T_e}(\boldsymbol{\mathcal{S}} \times \boldsymbol{\mathcal{S}_w}) = \mathbb{R}^{\boldsymbol{m}}$

Tool 2 : extension of the image of a diffeomorphism

Assumption : $\mathcal{X}$ bounded such that $\boldsymbol{x}(\boldsymbol{t}) \in \boldsymbol{\mathcal{X}}$ for all t and any x_0 in $\mathcal{X}_0$

Theorem (Observer with coordinate augmentation)

*Assume there exist a **diffeomorphism** $T_e : \mathcal{S}_a \to \mathbb{R}^m$ such that*

1. $T_e(x,0) \;=\; T(x) \qquad \forall x \in \mathcal{X}$ *=> Problem 1*
2. $T_e(\mathcal{S}_a) \;=\; \mathbb{R}^m$ *=> Problem 2*

For any initial condition x_0 in $\mathcal{X}_0$ and $(\hat{x}_0, \hat{w}_0)$ in $\mathcal{S}_a$, the solution of

$$\dot{\overbrace{\begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix}}} = \left(\frac{dT_e}{d(x,w)}(\hat{x},\hat{w}) \right)^{-1} \underbrace{\varphi(T_e(\hat{x},\hat{w}),u,y)}_{\text{given observer}}$$

is defined on $[0,+\infty)$ and satisfies :

$$\lim_{t\to+\infty} |\hat{w}(t)| + |x(t)-\hat{x}(t)| = 0 \; .$$

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Sufficient condition to solve Problem 1

Theorem (Jacobian completion)

Let $T : \mathcal{S} \subset \mathbb{R}^n \to \mathbb{R}^m$ *be an injective immersion.*

If there exist

- *a* ***bounded*** *open set* $\tilde{\mathcal{S}}$ *such that* $\mathrm{cl}(\tilde{\mathcal{S}}) \subset \mathcal{S}$
- *a* C^2 *function* $\gamma : \mathrm{cl}(\tilde{\mathcal{S}}) \to \mathbb{R}^{m*(m-n)}$ *satisfying :*

$$\underbrace{\det\left(\frac{dT}{dx}(x) \quad \gamma(x)\right) \neq 0}_{\textbf{continuous} \text{ completion of the Jacobian}} \qquad \forall x \in \mathrm{cl}(\tilde{\mathcal{S}}) \,,$$

then there exists $\varepsilon > 0$ *such that the function defined by*

$$T_a(x, w) \;=\; T(x) \;+\; \gamma(x)\, w$$

is a diffeomorphism on

$$\mathcal{S}_a = \tilde{\mathcal{S}} \times \mathcal{B}_\varepsilon(0) \subset \mathbb{R}^{m-n}$$

How to complete continuously a full-rank matrix ?

Wazewski's approach

Wazewski's problem :

Given mn continuous functions $\psi_{ij} : \mathcal{S} \subset \mathbb{R}^n \to \mathbb{R}$, *can we find* $m(m-n)$ *continuous functions* $\gamma_{kl} : \mathcal{S} \to \mathbb{R}$ *such that the matrix*

$$P(x) = \begin{pmatrix} \psi(x) & \gamma(x) \end{pmatrix}$$

is invertible for all x *in* $\mathcal{S}$ *?*

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Théorème (Wazewski)

If $\mathcal{S} \subset \mathbb{R}^n$ *is* ***contractible****, then there exists a* C^∞ *function* γ *making the matrix* $P(x)$ *invertible for all* x *in* $\mathcal{S}$*.*

But are there cases where an explicit and universal formula exists ?

Example

$$T(x) = \begin{pmatrix} \psi_1(x_1, x_2) \\ \psi_2(x_1, x_2) \\ \psi_3(x_1, x_2) \end{pmatrix} \quad \longrightarrow \quad \frac{dT}{dx}(x) = \begin{pmatrix} \partial_1\psi_1(x_1, x_2) & \partial_2\psi_1(x_1, x_2) \\ \partial_1\psi_2(x_1, x_2) & \partial_2\psi_2(x_1, x_2) \\ \partial_1\psi_3(x_1, x_2) & \partial_2\psi_3(x_1, x_2) \end{pmatrix} \quad \text{full rank}$$

How to add $m - n = 1$ column to get an invertible matrix for all $x \in \mathcal{S}$?

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$$\det \begin{pmatrix} \partial_1\psi_1(x_1,x_2) & \partial_2\psi_1(x_1,x_2) & M_{13}(x) \\ \partial_1\psi_2(x_1,x_2) & \partial_2\psi_2(x_1,x_2) & -M_{23}(x) \\ \partial_1\psi_3(x_1,x_2) & \partial_2\psi_3(x_1,x_2) & M_{33}(x) \end{pmatrix} = M_{13}(x)^2 + M_{23}(x)^2 + M_{33}(x)^2 \neq 0$$

where $M_{ij}(x)$ are the matrix minors.

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=> not always possible for topological reasons

Eckmann's approach

$P[m,n]$ problem :

For a $m \times n$ *continuous matrix* ψ *of rank* n *(with* $n < m$*), can we find* $\gamma(\psi)$ *of dimension* $m \times m-n$ *continuous in the coefficients of* ψ*, such that :*

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Theorem (Eckmann)

The $P[m, n]$ problem has a solution only in the following cases :

$m =$	≥ 2	4	8
$n =$	$m - 1$	1	1

$\Rightarrow$ always possible to have $n = m - 1$, maybe adding $m - n - 1$ zeros to T

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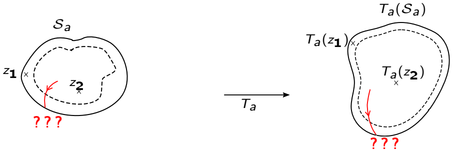
$z = (x, w)$



$$\dot{z} = \left(\frac{\partial T_a}{\partial z}(\hat{z})\right)^{-1} \varphi(T_a(\hat{z}), u, y) \quad \xleftarrow[T_a^{-1}]{} \quad \dot{\hat{\xi}} = \varphi(\hat{\xi}, u, y)$$

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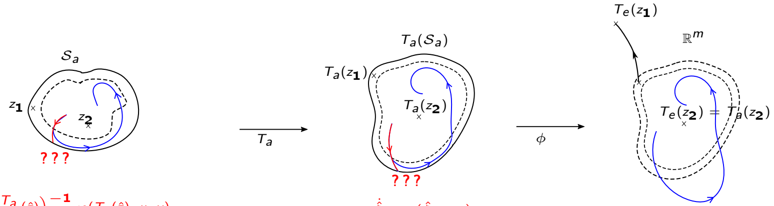

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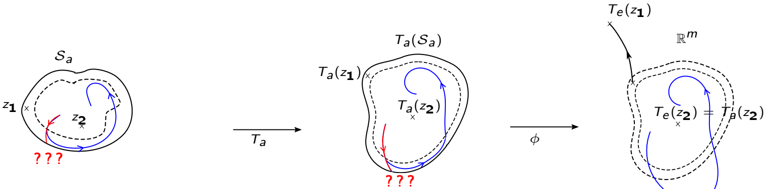
$$\dot z = \left(\frac{\partial T_e}{\partial z}(\hat z)\right)^{-1} \varphi(T_e(\hat z), u, y) \quad \xleftarrow{\;T_e^{-1} = (\phi \circ T_a)^{-1}\;} \quad \dot{\hat\xi} = \varphi(\hat\xi, u, y)$$

$z = (x, w)$



$\dot{z} = \left(\frac{\partial T_a}{\partial z}(\hat{z})\right)^{-1} \varphi(T_a(\hat{z}), u, y)$ $\xleftarrow{T_a^{-1}}$ $\dot{\hat{\xi}} = \varphi(\hat{\xi}, u, y)$

$\dot{z} = \left(\frac{\partial T_e}{\partial z}(\hat{z})\right)^{-1} \varphi(T_e(\hat{z}), u, y)$ $\xleftarrow{T_e^{-1} = (\phi \circ T_a)^{-1}}$ $\dot{\hat{\xi}} = \varphi(\hat{\xi}, u, y)$

1) Look for ϕ diffeomorphism from $T_a(\mathcal{S}_a)$ to $\mathbb{R}^m$

2) Take $T_e = \phi \circ T_a$

=> $T_e(\mathcal{S}_a) = \mathbb{R}^m$

=> completeness of solutions

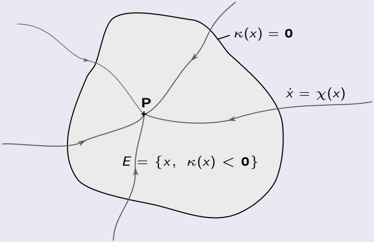
Sufficient conditions for image extension

Condition $(\mathcal{C})$

An open subset E of $\mathbb{R}^m$ verifies Condition $(\mathcal{C})$ if there exist a C^1 function $\kappa : \mathbb{R}^m \to \mathbb{R}$, a bounded C^1 vector field χ, and a point P in E such that :

- $E = \{\xi \in \mathbb{R}^m \;:\; \kappa(\xi) < 0\}$
- P is globally attractive for χ
- we have the transversality property :

$$\frac{\partial \kappa}{\partial \xi}(\xi)\chi(\xi) < 0 \qquad \forall \xi \in \mathbb{R}^m : \kappa(\xi) = 0.$$



Sufficient conditions for image extension :

Theorem (Image extension)

Let $\mathcal{S}_a$ be an open subset of $\mathbb{R}^m$ and $T_a : \mathcal{S}_a \to \mathbb{R}^m$ be a diffeomorphism.

If

either $T_a(\mathcal{S}_a)$ *verifies Condition* $(\mathcal{C})$,

then there exists a diffeomorphism $T_e : \mathcal{S}_a \to \mathbb{R}^m$ *such that*

- $T_e = T_a$ *on* $\mathcal{X} \times \{0\}$
- $T_e(\mathcal{S}_a) = \mathbb{R}^m$

i-e Problem 2 is solved.

Sufficient conditions for image extension :

Theorem (Image extension)

Let $\mathcal{S}_a$ be an open subset of $\mathbb{R}^m$ and $T_a : \mathcal{S}_a \to \mathbb{R}^m$ be a diffeomorphism.

If

either $T_a(\mathcal{S}_a)$ *verifies Condition* $(\mathcal{C})$,

or $\mathcal{S}_a$ *is* C^2*-diffeomorphic to* $\mathbb{R}^m$ *and* T_a *is* C^2,

then there exists a diffeomorphism $T_e : \mathcal{S}_a \to \mathbb{R}^m$ *such that*

- $T_e = T_a$ *on* $\mathcal{X} \times \{0\}$
- $T_e(\mathcal{S}_a) = \mathbb{R}^m$

i-e Problem 2 is solved.

Contents

Conclusion

Goal : augmentation of injective immersion into surjective diffeomorphism

Application :

- observer implementation in the plant coordinates
- numerical inversion of an injective immersion
- no local minima and completeness of solutions guaranteed

Difficulty : practical implementation of the image extension