An elapsed time model for strongly coupled inhibitory and excitatory neural networks.

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Background

- Modeling the brain is an interesting problem from both mathematical and neuroscience viewpoint. PDEs arising from stochastic models have been a very active area in recent decades.

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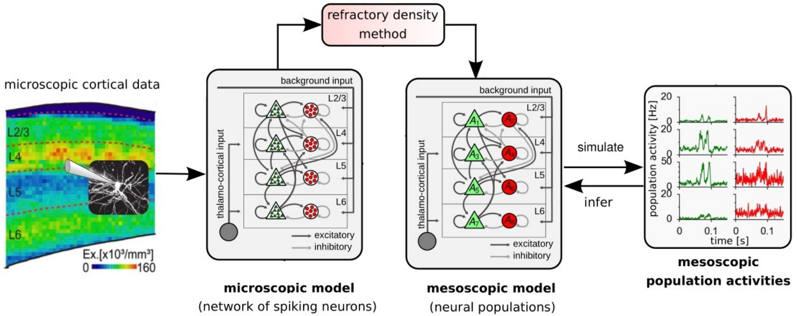
Background

- Modeling the brain is an interesting problem from both mathematical and neuroscience viewpoint. PDEs arising from stochastic models have been a very active area in recent decades.
- One of these equations is the **elapsed-time model**, where neurons are described by their elapsed time since their last discharge i.e. the **refractory period**.
- Neurons undergo a charging process and then a sudden discharge takes place in response to certain stimulus and this causes other neighboring neurons to discharge depending on the intensity of interconnections.

Background



microscopic cortical data
L2/3
L4
L5
L6
Ex.[x10³/mm³]
0
160
refractory density method
background input
thalamo-cortical input
excitatory
inhibitory
microscopic model
(network of spiking neurons)
mesoscopic model
(neural populations)
simulate
infer
population activity [Hz]
time [s]
mesoscopic population activities

Elapsed time model

Neural network modelled via the age-structure equation

$$\begin{cases} \partial_t n(t,s) + \partial_s n(t,s) + p(s,N(t))n(t,s) = 0, \\ N(t) := n(t,s=0) = \int_0^\infty p(s,N(t))n(t,s)\,ds, \\ n(t=0,s) = n_0(s). \end{cases} \tag{1}$$

- $n(t,s)$: Density of neurons at time t, whose elapsed time since the last discharge is s.
- $N(t)$: **Flux of discharging neurons**, which corresponds to the neural activity.

Elapsed time model

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- $p(s,u)$: **Firing rate** of discharging neurons of age s, submitted to an amplitude of stimulation $u \geq 0$.
- Inhibitory case: $\partial_u p < 0$.
- Excitatory case: $\partial_u p > 0$.
- Weak interconnections: $\|\partial_u p\|_\infty$ small.

Elapsed time model

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- $n_0(s)$: Initial data, with $n_0 \in L^1$ and $n_0 \geq 0$.
- Mass-conservation:

$$\int_0^\infty n(t,s)\,ds = \int_0^\infty n_0(s)\,ds = 1 \quad \forall t > 0.$$

Steady states

Time-independent problem for (n^*, N^*)

$$\begin{cases} \partial_s n^*(s) + p(s, N^*) n^*(s) = 0, \\ N^* := n(s=0) = \int_0^\infty p(s, N^*) n^*(s)\, ds, . \end{cases} \tag{2}$$

Solved through the formula

$$n^*(s) = N^* e^{-\int_0^s p(s', N^*) ds'},$$
$$N^* = F(N^*) := \left(\int_0^\infty e^{-\int_0^u p(s', N^*) ds'} du \right)^{-1}.$$

Unique solution for weak interconnections and inhibitory case.

Convergence for the weak interconnections case

Theorem (Cañizo-Yoldas)

Assume that $p \in L^\infty$ *satisfies for some constant* $p_0, p_\infty, \sigma > 0$

$$p_0 \mathbb{1}_{\{s>\sigma\}} \leq p(s,u) \leq p_\infty.$$

Then for weak interconnections there exists $C, \lambda > 0$ *such that the solution* n *of* (1) *satisfies for all* $t > 0$

$$\|n(t) - n^*\|_{L^1} \leq Ce^{-\lambda t}\|n_0 - n^*\|_{L^1}.$$

Moreover $|N(t) - N^*|$ *converges exponentially to* 0 *when* $t \to \infty$.

Dynamics in the general case?

Classical methods like Entropy method and Doeblin's theory deal only with weak interconnections. The main idea is to consider a case beyond weak interconnections.

Conjecture (Strong interconnections)

Assume $\|p\|_\infty$ *large enough. If* $\partial_u p < 0$*, then there all the solutions converge to a unique steady state. If* $\partial_u > 0$*, then a periodic solution may arise.*

Assumptions on the firing rate

Consider p given by

$$p(s,u) = \varphi(u)\mathbb{1}_{\{s>\sigma\}},$$

with $\sigma > 0$ and φ smooth satisfying

$$0 < \alpha \leq \varphi(u) \leq \beta.$$

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Important function for the analysis

$$\psi(u) := \frac{u}{\varphi(u)} \quad \text{with} \quad \psi'(u) = \frac{\varphi(u) - u\varphi'(u)}{\varphi^2(u)}.$$

Cases to study: ψ' strictly positive and ψ' changing sign.

Reduction to a delay differential equation

Lemma (Reduction to a delay equation)

For $t > \sigma$ the discharge flux $N(t)$ satisfies

$$\int_{t-\sigma}^{t} N(s)\, ds + \psi(N(t)) = 1. \tag{3}$$

Moreover if $N(t)$ is smooth for $t > \sigma$, the following formula for $N'(t)$ holds

$$\frac{d}{dt}\psi(N(t)) = N(t-\sigma) - N(t). \tag{4}$$

Reconstructing a solution from a given activity

Theorem (Reconstruction theorem)

Let $N \in L^\infty(0,\infty)$ *a non-negative satisfying* $\psi(N(t)) \in \mathcal{C}([\sigma,\infty)) \cap \mathcal{C}^1((0,\sigma))$ *and the following conditions:*

$$\bullet N(\sigma - t) + \frac{d(\psi(N))}{dt}(\sigma - t) \geq 0 \quad for \quad 0 < t < \sigma.$$

$$\bullet N(t) \; solves \; the \; integral \; equation \; (3) \; for \quad t \geq \sigma.$$

Then we can construct a solution of elapsed-time equation with $N(t)$ *as activity.*

Convergence for the inhibitory and weakly excitatory case

Theorem (Convergence for $\psi' > 0$)

Assume that $\psi' > 0$, then the solutions of the system (1) *satisfy*

$$\|n(t) - n^*\|_{L^1} \to 0, \quad |N(t) - N^*| \to 0, \quad when \quad t \to \infty,$$

and $N(t)$ oscillates around N^*.

Convergence for the inhibitory and weakly excitatory case

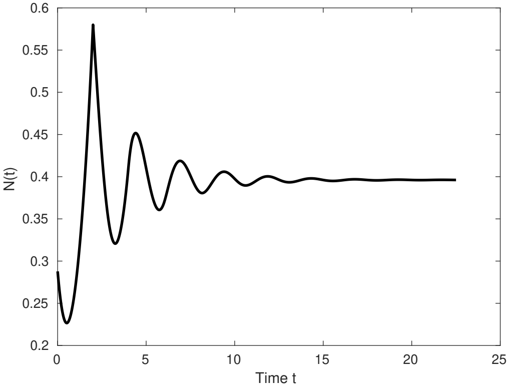


Figure: Activity $N(t)$ for $\varphi(u) = 1 + 2e^{-2u}$, $\sigma = 2$ and $n_0(s) = e^{-s}$.

Convergence for the inhibitory and weakly excitatory case

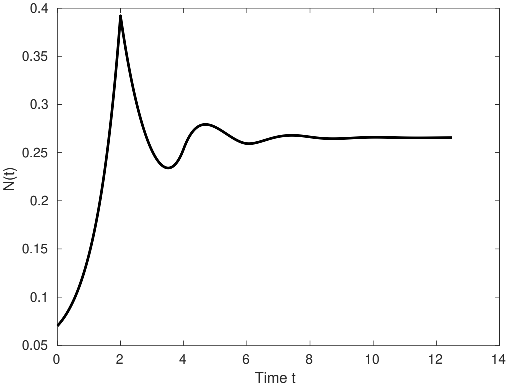


Figure: Activity $N(t)$ for $\varphi(u) = (1 + e^{-u})^{-1}$, $\sigma = 2$ and $n_0(s) = e^{-s}$.

Periodic solutions when ψ' changes sign

Theorem (Existence of periodic solutions)

Consider ψ *smooth and* $0 < \bar{N}^- < \underline{N} < \bar{N}^+$, *such that*

$$\psi(\bar{N}^-) = \psi(\bar{N}^+)$$

with a unique local minimum $\underline{N} \in (\bar{N}^-, \bar{N}^+)$.

Then for σ *small there exists a* 2σ *periodic solution* $N(t)$ *of the delay equation with* $\psi(N) \in W^{1,\infty}(\mathbb{R})$ *and the following conditions:*

1. $\begin{cases} \underline{N} < N(t) < \bar{N}^+, \; N'(t) < 0 \quad \textit{for} \quad t \in (0, \sigma), \\ \bar{N}^- < N(t) < \underline{N}, \; N'(t) < 0 \quad \textit{for} \quad t \in (\sigma, 2\sigma). \end{cases}$
2. $N(0^+) = \bar{N}^+$, $N(2\sigma^-) = \bar{N}^-$.
3. $\psi(N(\sigma^-)) = \psi(N(\sigma^+))$.

Periodic solutions when ψ' changes sign

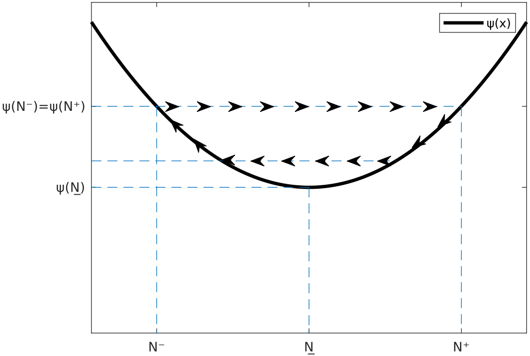


Figure: Example of values $\bar{N}^- < \underline{\mathsf{N}} < \bar{N}^+$ satisfying the hypothesis.

Periodic solution when ψ' changes sign

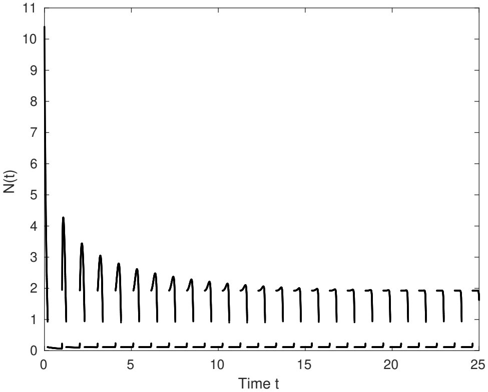


Figure: Activity $N(t)$ for $\varphi(u) = \frac{10u^2}{u^2+1} + 0.5$, $\sigma = 1$ and $n_0(s) = e^{-(s-1)}\mathbb{1}_{\{s>1\}}$.

Periodic solution when ψ' changes sign

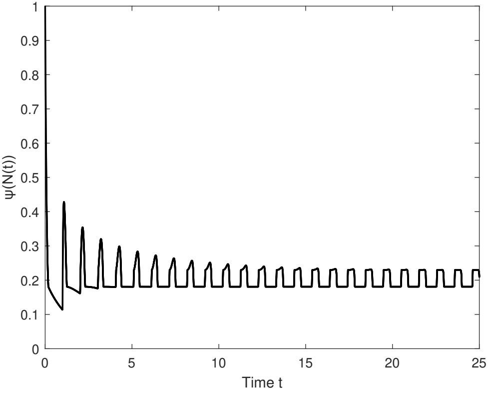


Figure: Plot of $\psi(N(t))$ for $\varphi(u) = \frac{10u^2}{u^2+1} + 0.5$, $\sigma = 1$ and $n_0(s) = e^{-(s-1)}\mathbb{1}_{\{s>1\}}$.

Multiplicity of solutions when ψ' changes sing

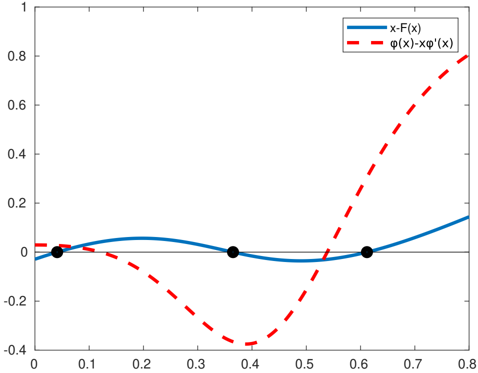
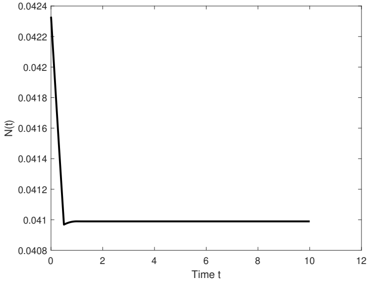


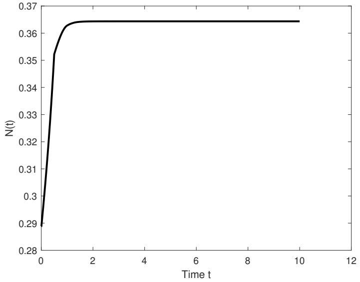
Figure: Multiple equilibriums for $\varphi(N) = \frac{1}{1+e^{-9N+3.5}}$ and $\sigma = 0.5$.

Multiplicity of solutions when ψ' changes sing

Figure: Different possible solutions for $n_0(s) = e^{-(s-0.5)}\mathbb{1}_{\{s>0.5\}}$



(a) Activity $N(t)$ for N_0^1.



(b) Activity $N(t)$ for N_0^2.

Multiplicity of solutions when ψ' changes sing

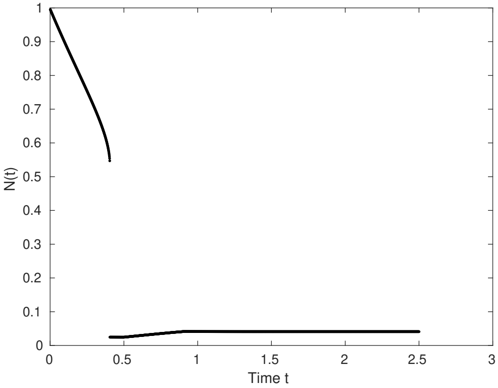


Figure: Activity $N(t)$ for N_0^3.

Open questions

- Is the type of obtained solutions structurally stable?
- Speed of convergence to the observed patterns?
- When a solution has jump discontinuities?
- What kind of periodic patterns are attractive?
- Existence of continuous periodic solutions?

Thank you for your attention.