Non-overlapping domain decomposition methods with non-local transmission conditions for time harmonic wave propagation problems

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conducted in the INRIA team POEMS

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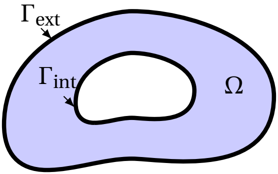
Domain decomposition with non-local transmission operators

Model problem

$$\begin{cases} (-\Delta - k^2)\,u = f & \text{in } \Omega \\ \left(k^{-1}\,\partial_{\mathbf{n}} - \mathrm{i}\right) u = 0 & \text{on } \Gamma_{\text{ext}} \\ \partial_{\mathbf{n}} u = 0 & \text{on } \Gamma_{\text{int}} \end{cases}$$



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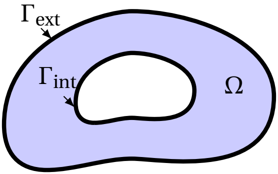


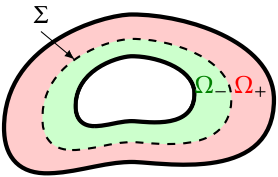
Can be generalized to

- variable coefficients
- other type of boundary conditions

Domain decomposition

$$\begin{cases}(-\Delta - k^2)u_- = f|_{\Omega_-}\\ \text{T.C. on } \Sigma\end{cases}$$

$$\begin{cases}(-\Delta - k^2)u_+ = f|_{\Omega_+}\\ \text{T.C. on } \Sigma\end{cases}$$



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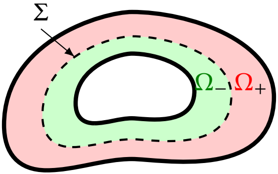

"Onion-like" decomposition
⇒ no cross-points [in the first part of the talk]

Domain decomposition

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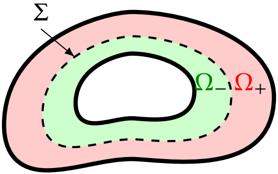


Which choice of transmission condition on Σ?

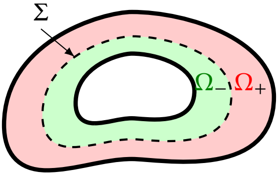
- Continuity of traces ($f \in L^2(\Omega)$)

$$\begin{cases}\gamma_0\, u_- = \gamma_0\, u_+\\ \gamma_1\, u_- = \gamma_1\, u_+\end{cases} \quad \text{where} \quad \begin{cases}\gamma_0\, u := u|_\Sigma & \text{Dirichlet}\\ \gamma_1\, u := k^{-1}\,\partial_{\mathbf{n}} u|_\Sigma & \text{Neumann}\end{cases}$$

Transmission operator

$$\begin{cases}(-\Delta - k^2)u_- = f|_{\Omega_-}\\ \left(\gamma_1 + \mathrm{i}\mathrm{T}\gamma_0\right)\, u_- = \left(\gamma_1 + \mathrm{i}\mathrm{T}\gamma_0\right)\, u_+\end{cases}$$

$$\begin{cases}(-\Delta - k^2)u_+ = f|_{\Omega_+}\\ \left(\gamma_1 - \mathrm{i}\mathrm{T}\gamma_0\right)\, u_+ = \left(\gamma_1 - \mathrm{i}\mathrm{T}\gamma_0\right)\, u_-\end{cases}$$



Impedance-based / generalized Robin transmission condition

- T is a boundary operator

$$\begin{cases}\gamma_0\, u_- = \gamma_0\, u_+\\ \gamma_1\, u_- = \gamma_1\, u_+\end{cases} \quad \Leftrightarrow \quad \begin{cases}\left(\gamma_1 + \mathrm{i}\mathrm{T}\,\gamma_0\right)\, u_- = \left(\gamma_1 + \mathrm{i}\mathrm{T}\,\gamma_0\right)\, u_+\\ \left(\gamma_1 - \mathrm{i}\mathrm{T}\,\gamma_0\right)\, u_- = \left(\gamma_1 - \mathrm{i}\mathrm{T}\,\gamma_0\right)\, u_+\end{cases}$$

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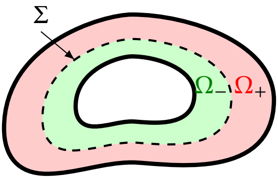

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The equivalence holds provided T is injective

Transmission operator

$$\begin{cases} (-\Delta - k^2) u_- = f|_{\Omega_-} \\ \left(\gamma_1 + \mathrm{i} \mathrm{T} \gamma_0\right) u_- = \left(\gamma_1 + \mathrm{i} \mathrm{T} \gamma_0\right) u_+ \end{cases}$$

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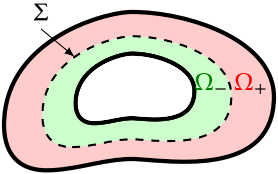

Sufficient condition for **well-posedness** of local problems

- T is a positive and self-adjoint boundary operator

Proof: Fredholm alternative

Reformulation at the interface

$$\begin{cases} (-\Delta - k^2)u_- = f|_{\Omega_-} \\ (\gamma_1 + \mathrm{i}\mathrm{T}\gamma_0)\, u_- = (\gamma_1 + \mathrm{i}\mathrm{T}\gamma_0)\, u_+ \end{cases}$$

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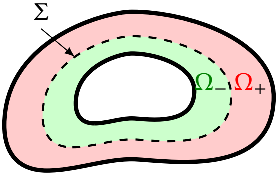

- Reformulation at the interface

$$\mathbf{x} := \begin{bmatrix} x_- \\ x_+ \end{bmatrix} := \begin{bmatrix} (\gamma_1 + \mathrm{i}\mathrm{T}\,\gamma_0)\, u_- \\ (\gamma_1 - \mathrm{i}\mathrm{T}\,\gamma_0)\, u_+ \end{bmatrix} \qquad \Bigg(\mathrm{Id} - \underbrace{\begin{bmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{bmatrix}}_{\Pi} \underbrace{\begin{bmatrix} \mathrm{S}_- & 0 \\ 0 & \mathrm{S}_+ \end{bmatrix}}_{\mathbf{S}} \Bigg) \begin{bmatrix} x_- \\ x_+ \end{bmatrix} = \mathbf{b}$$

$$(\mathrm{Id} - \mathbf{\Pi S})\,\mathbf{x} = \mathbf{b}$$

Reformulation at the interface

$$\begin{cases}(-\Delta - k^2)u_-^n = f|_{\Omega_-}\\ (\gamma_1 + \mathrm{i}T\gamma_0)\ u_-^n = (\gamma_1 + \mathrm{i}T\gamma_0)\left[r\,u_-^{n-1} + (1-r)\,u_+^{n-1}\right]\end{cases}$$

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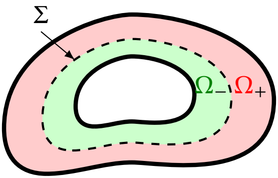

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$$(\mathrm{Id} - \mathbf{\Pi S})\,\mathbf{x} = \mathbf{b}$$

Analysis: (relaxed) Jacobi algorithm

$$\mathbf{x}^n = r\,\mathbf{x}^{n-1} + (1-r)\mathbf{\Pi S}\,\mathbf{x}^{n-1} + \mathbf{b} \qquad n \in \mathbb{N}$$

Convergence result

$$\begin{cases} (-\Delta - k^2)u_-^n = f|_{\Omega_-} \\ (\gamma_1 + \mathrm{i}\mathrm{T}\gamma_0)\; u_-^n = (\gamma_1 + \mathrm{i}\mathrm{T}\gamma_0)\left[r\,u_-^{n-1} + (1-r)\,u_+^{n-1}\right] \end{cases}$$

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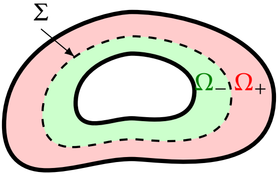

THEOREM If $r \in (0,1)$ and T is a positive self-adjoint isomorphism

$$\mathrm{T} : H^{1/2}(\Sigma) \longrightarrow H^{-1/2}(\Sigma)$$

then the iterative algorithm converges **geometrically**

$$\exists\, \tau < 1 : \qquad \left\| u_+^n - u_+ \right\| + \left\| u_-^n - u_- \right\| \leq C\,\tau^n$$

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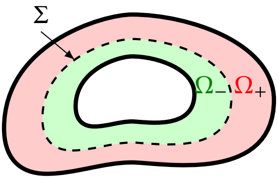

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Remark: T is necessarily non-local

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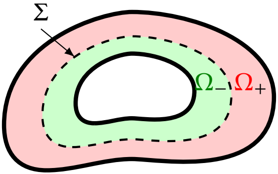

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Proof: [**Collino Ghanemi Joly** 2000] [**Collino Joly Lecouvez** 2020]

- **ΠS** is a contraction (energy conservation result)
- Id – **ΠS** is an isomorphism (well-posedness of a transmission problem)

Convergence result

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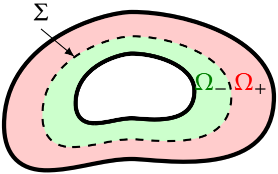

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Analysis on the (relaxed) Jacobi algorithm

- GMRES algorithm in practice
- The GMRES solution will satisfy the same type of bound

Convergence result

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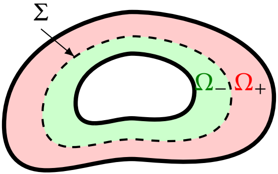

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Stable convergence at the discrete level [**Claeys Collino Joly P.** 2019]

- if $\langle \mathrm{T}\cdot,\cdot\rangle$ satisfies a uniform discrete inf-sup condition
- C and τ independent of the discretization parameter h

Convergence result

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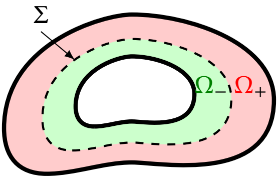

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Can be extended to the electromagnetic setting

$$\begin{cases}\gamma_0\, \mathbf{u} := \mathbf{n} \times (\mathbf{u} \times \mathbf{n}) & \in H_{\mathrm{curl}}^{-1/2}(\Sigma)\\ \gamma_1\, \mathbf{u} := k^{-1}\, \mathbf{n} \times \mathbf{curl}\, \mathbf{u} & \in H_{\mathrm{div}}^{-1/2}(\Sigma)\end{cases}$$

Choice of transmission operator T

Zeroth-order operators ($\alpha \,\mathrm{Id}$)

- convergence theory
- algebraic convergence

  $\|u_+^n - u_+\| + \|u_-^n - u_-\| \le C\, n^{-p},\ p > 0$

Helmholtz:
Després 1991
Maxwell:
Després Joly Roberts 1992

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Second-order operators ($\alpha \, \mathrm{Id} - \beta \Delta_\Sigma$)
(including rational fractions)

- very efficient in practice
- no general analysis

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Gander Magoules Nataf 2002
Boubendir Antoine Geuzaine 2012
Després Nicolopoulos Thierry 2020
Maxwell:
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Dolean Gander Gerardo-Giorda 2009
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Non-local operators

- complete analysis
- geometric convergence
  $\|u_+^n - u_+\| + \|u_-^n - u_-\| \leq C\,\tau^n,\ \tau < 1$
- stable after discretization

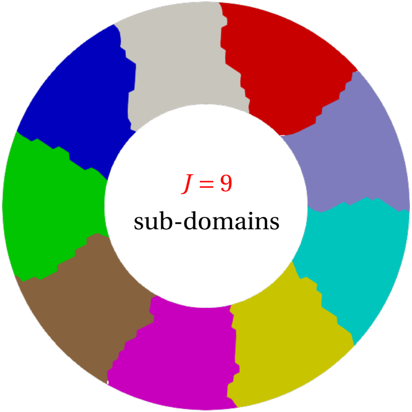
Helmholtz:
Ghanemi 1996
Collino Ghanemi Joly 1998
Lecouvez 2015
Collino Joly Lecouvez 2020
Claeys Collino Joly P. 2020
Claeys P. 2020
Maxwell:
Claeys Thierry Collino 2017
P. 2020

Dealing with cross-points

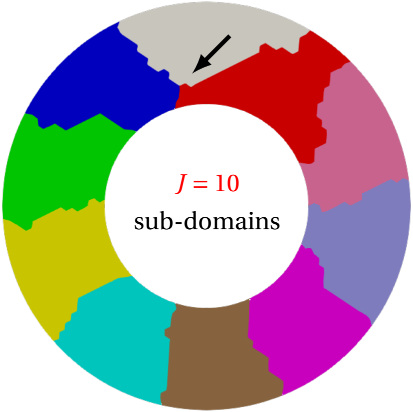
The cross-point issue: a motivating experiment

Non-local operators

- At the **continuous** level: the convergence proof fails [**Collino Ghanemi Joly** 2000]
- At the **discrete** level: unstable convergence

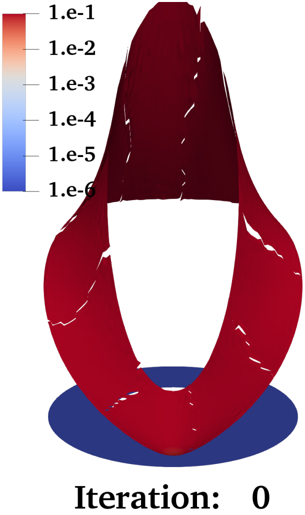


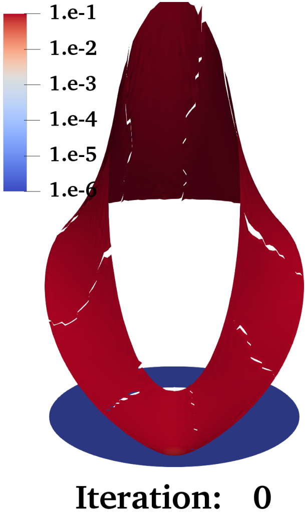
No cross-point



With cross-point

The cross-point issue: a motivating experiment

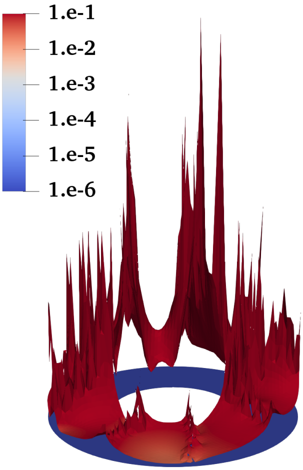




Local operator	Non-local operator

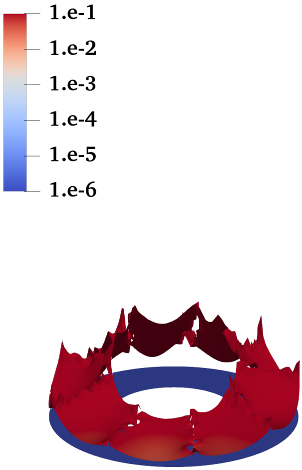
No cross-point — the elevation represent the absolute error of the modulus

The cross-point issue: a motivating experiment



Iteration: 5

Local operator



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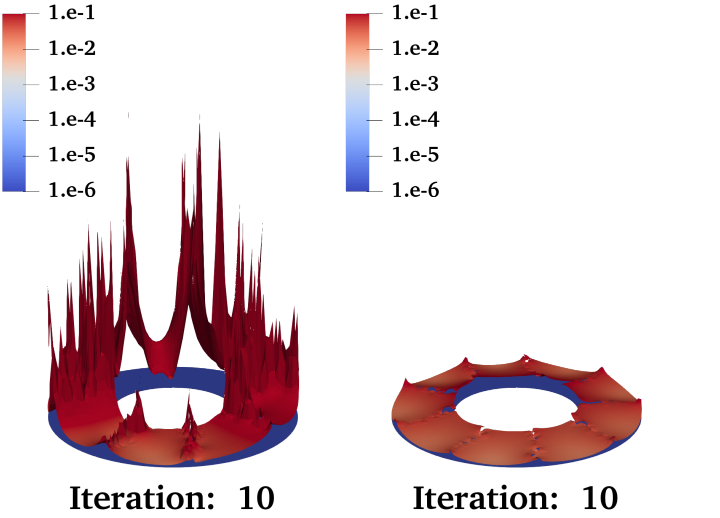

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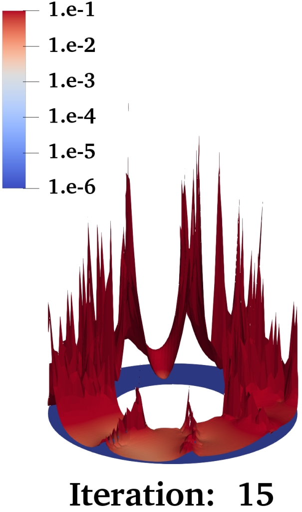

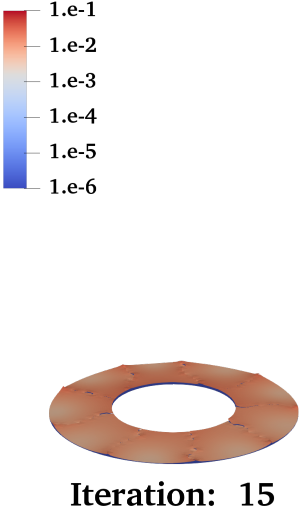

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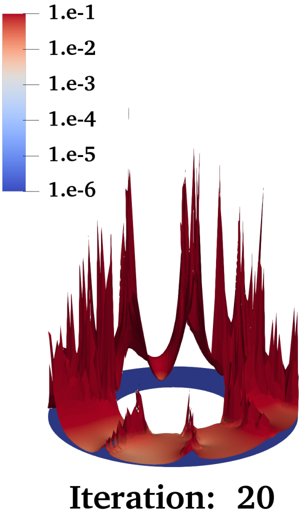
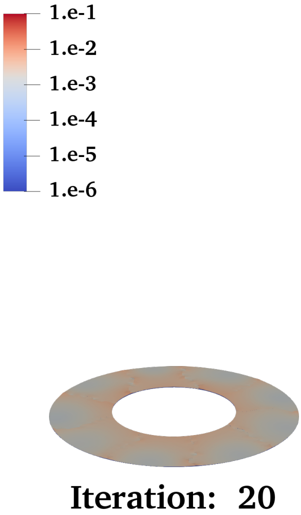
The cross-point issue: a motivating experiment





Local operator	Non-local operator

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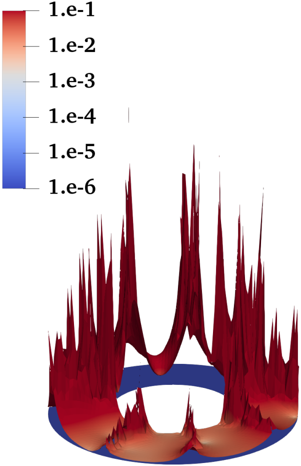
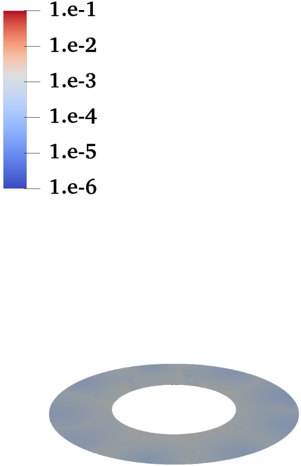


Iteration: 25

Local operator



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Non-local operator

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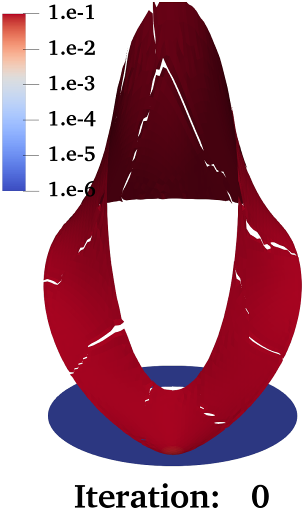

No treatment	With new treatment

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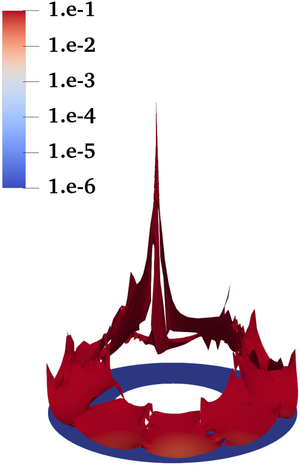

Iteration: 5

No treatment	With new treatment

With cross-point — the elevation represent the absolute error of the modulus

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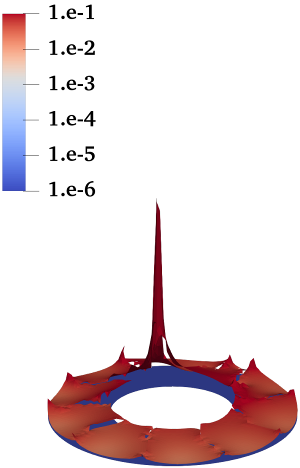

Iteration: 10

No treatment	With new treatment

With cross-point — the elevation represent the absolute error of the modulus

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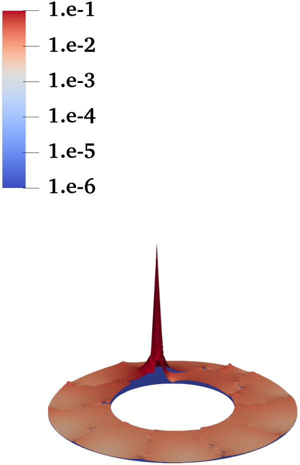

Iteration: 15

No treatment	With new treatment

With cross-point — the elevation represent the absolute error of the modulus

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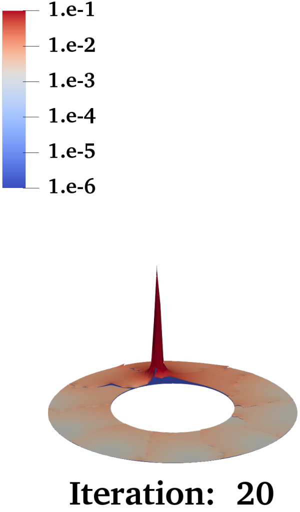

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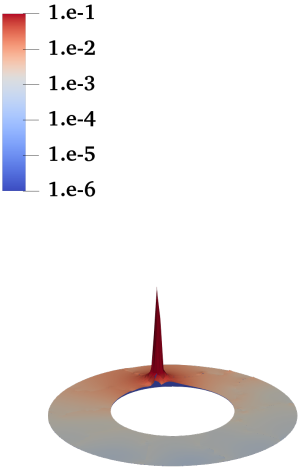

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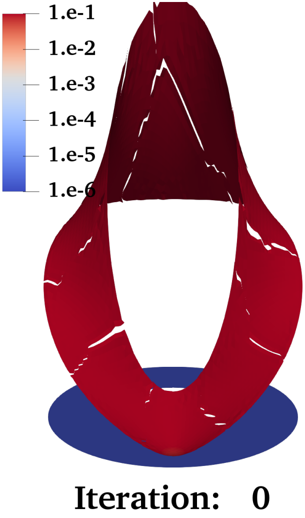

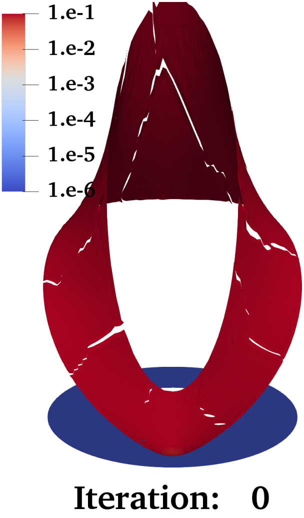

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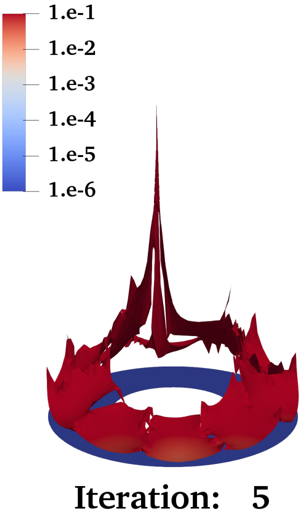

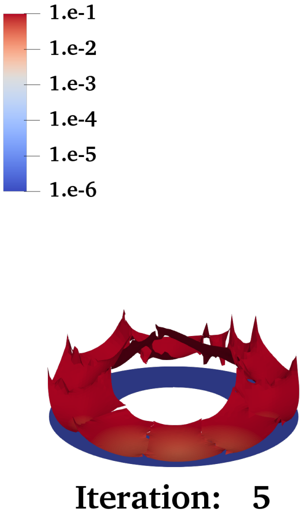

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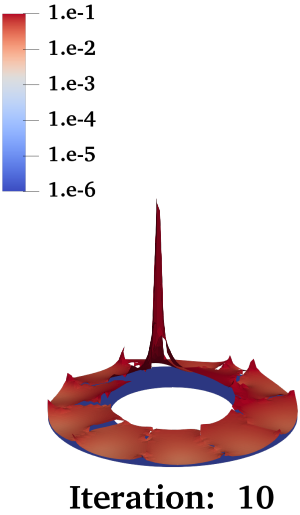

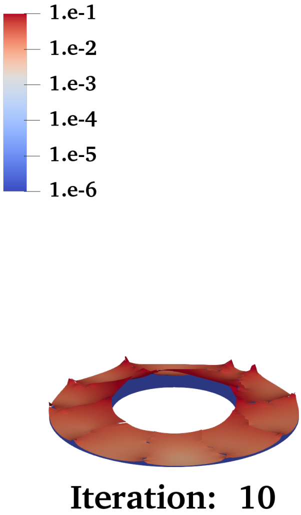

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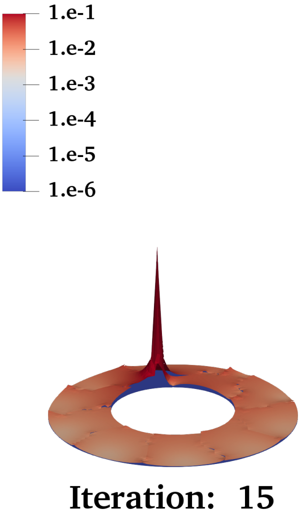

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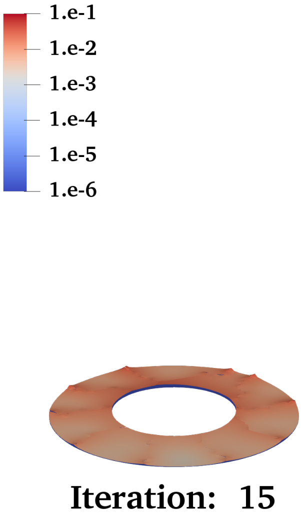

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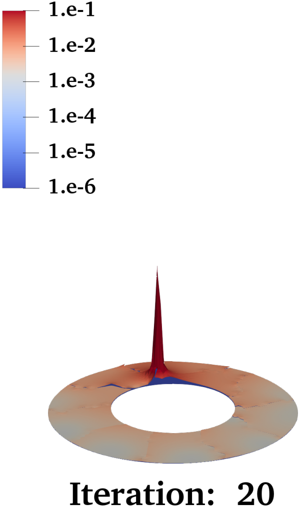

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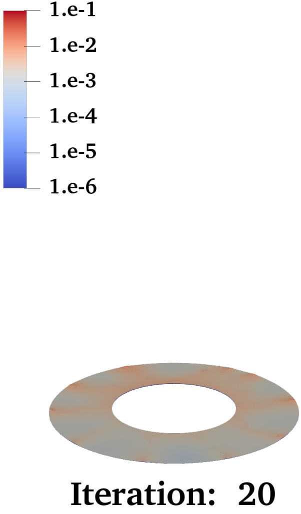

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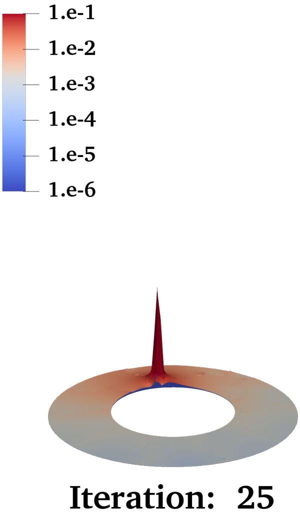

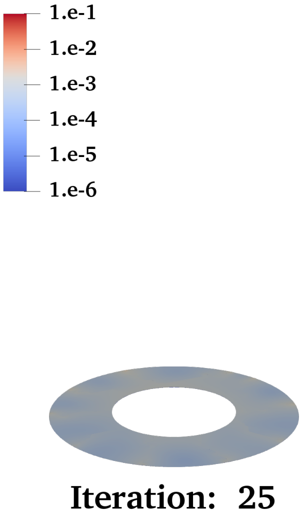

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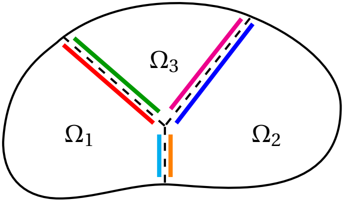
The Multi-Trace Formalism

Introduced in [**Claeys Hiptmair** 2013]

Paradigm shift: the traces are considered



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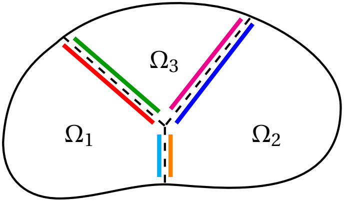

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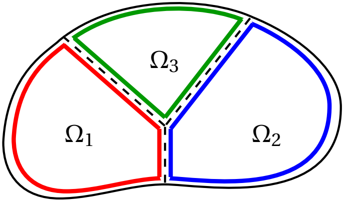

but rather at each sub-domain boundary

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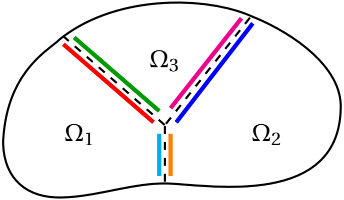

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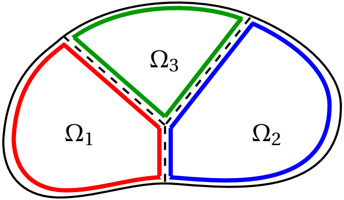

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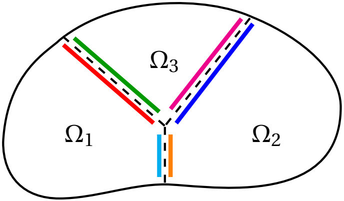

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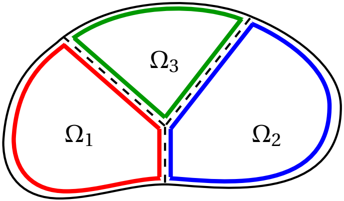

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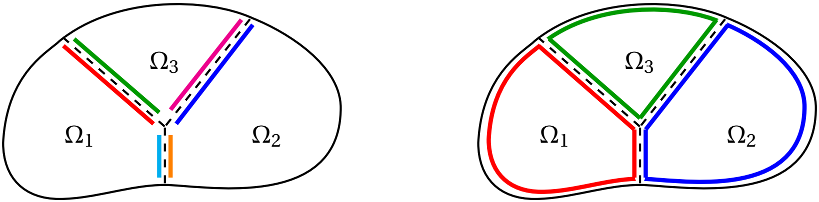

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- the skeleton problem can be solved in parallel with only neighboring sub-domains exchanging data

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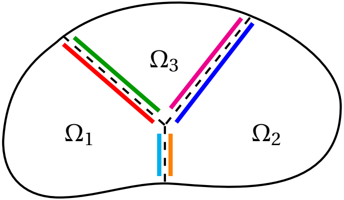

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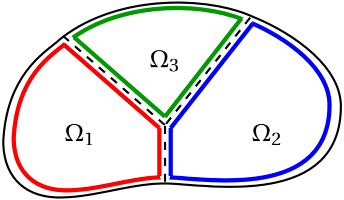

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$$(\mathrm{Id} - \mathbf{\Pi S})\,\mathbf{x} = \mathbf{b}$$

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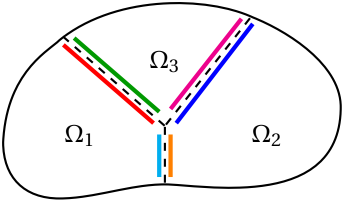

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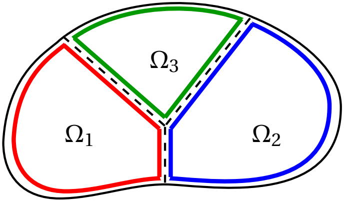

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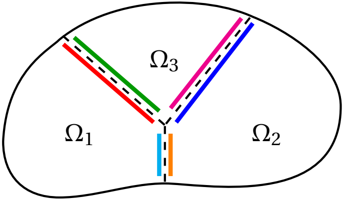

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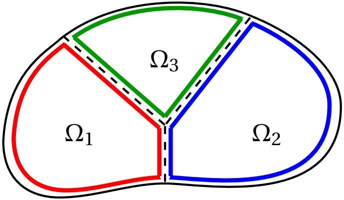

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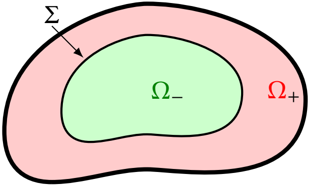
- complete convergence analysis: geometric rate, stability

Design of suitable non-local transmission operators

Definition of the transmission operator

Goal: construct a positive self-adjoint isomorphism such that

$$\mathrm{T} : H^{1/2}(\Sigma) \to H^{-1/2}(\Sigma)$$



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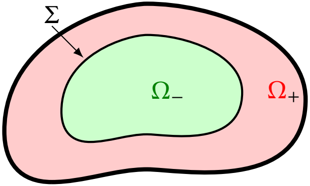
- Integral operators from potential theory



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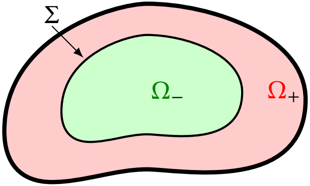

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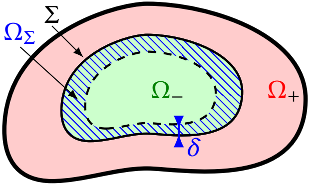

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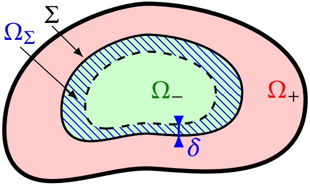

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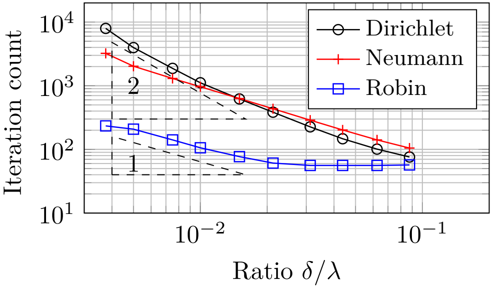
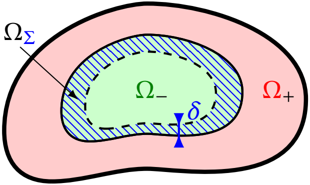
Advantages

- Easy to implement
- Lead to augmented but sparse linear systems
- Efficient even with varying coefficients, rough boundaries…

Influence of the truncation parameter δ





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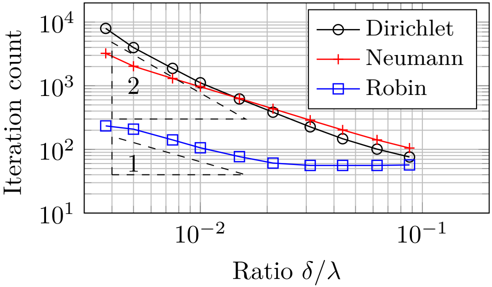

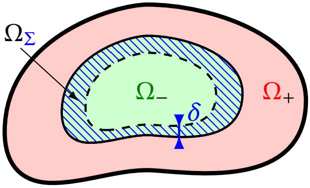

Fourier analysis

As $\delta \to 0$

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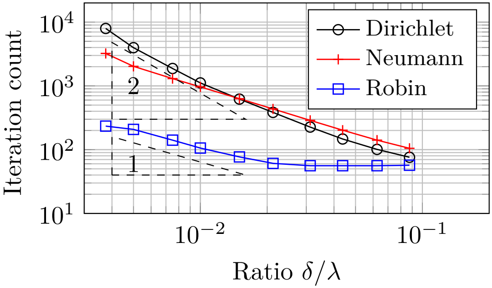

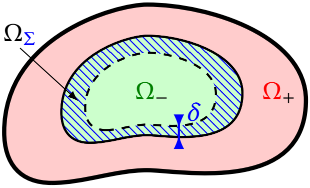

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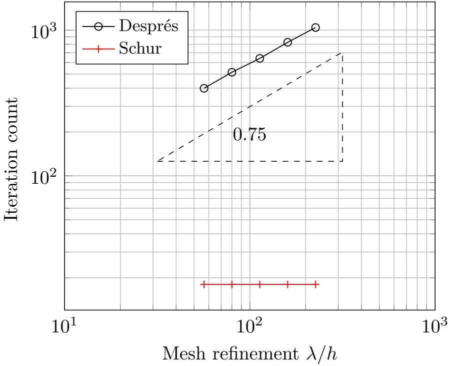
$\Rightarrow$ Controlled computational cost with maintained efficiency

Stability of the convergence



3D Maxwell problem — ball partitioned in 32 sub-domains — GMRES algorithm

⇒ **Stable** convergence when using non-local operators

Conclusions

Our approach for DDM for time harmonic wave propagation problems

- use non-local operators in transmission conditions

to have **theoretical guarantees** of

- geometric convergence — extension to Maxwell
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Thank you for your attention!