Large time asymptotics for evolution equations with mean field couplings

Jean Dolbeault

http://www.ceremade.dauphine.fr/~dolbeaul

Ceremade, Université Paris-Dauphine

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Outline

- **Nonlinear diffusion equations**
  ▷ The fast diffusion equation
  ▷ The subcritical Keller-Segel model
  ▷ A simple mean-field model

- **Hypocoercivity methods an overview**
  ▷ H^1, L^2, H^{-1}
  ▷ With several conservation laws

- **Vlasov-Fokker-Planck models**
  ▷ Hypocoercivity for the linear Vlasov-Fokker-Planck equation
  ▷ The Vlasov-Poisson-Fokker-Planck system: rates of convergence

- [bonus] **Decay and convergence rates for kinetic equations**

Entropy methods

Nonlinear diffusion equations

- ▷ The fast diffusion equation
- ▷ The subcritical Keller-Segel model
- ▷ A simple mean-field model

The fast diffusion equation

(Blanchet, Bonforte, JD, Grillo, Vázquez)
(Bonforte, JD, Nazaret, Simonov)

The fast diffusion equation

Consider the *fast diffusion* equation in $\mathbb{R}^d$, $d \geq 1$, $m < 1$

$$\frac{\partial u}{\partial t} = \Delta u^m \,, \quad u_{|t=0} = u_0 \geq 0 \tag{FDE}$$

With $p = \frac{1}{2\,m-1}$, $u = f^{2p}$

$$\frac{d}{dt} \underbrace{\int_{\mathbb{R}^d} u \, dx}_{=\|f\|_{2\,p}^{2p}} = 0 \,, \quad \frac{d}{dt} \underbrace{\int_{\mathbb{R}^d} u^m \, dx}_{=\|f\|_{p+1}^{p+1}} = (p+1)^2 \int_{\mathbb{R}^d} |\nabla f|^2 \, dx$$

Gagliardo-Nirenberg-Sobolev inequalities

$$\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \geq \mathcal{C}_{\rm GNS}(p) \, \|f\|_{2\,p} \tag{GNS}$$

$t \to +\infty$ asymptotics: $u(t,x) \sim B(t,x) = t^{-d/\mu} \, \mathsf{g}(t^{-1/\mu}\, x)^{2p}$
B Barenblatt self-similar solutions, $\mu = 2 - d\,(1-m) > 1$
$\mathsf{g}(x) = \left(1 + |x|^2\right)^{-\frac{1}{p-1}}$ Aubin-Talenti type function

Self-similar variables, entropy-entropy production inequality

In *self-similar variables* (FDE) becomes *a Fokker-Planck type equation*

$$\frac{\partial v}{\partial t} + \nabla \cdot \Big(v \left(\nabla v^{m-1} - 2\,x \right) \Big) = 0 \tag{1}$$

with (GNS) $\Longleftrightarrow \mathcal{I}[v] \geq 4\,\mathcal{F}[v]$ and $\frac{d}{dt}\mathcal{F}[v] = -\mathcal{I}[v]$

Generalized entropy (free energy) and *Fisher information*

$$\mathcal{F}[v] := \int_{\mathbb{R}^d} \left(\mathcal{B}^{m-1}\,(v-\mathcal{B}) - \frac{v^m - \mathcal{B}^m}{m} \right) dx\,, \quad \mathcal{I}[v] := \int_{\mathbb{R}^d} v\,\left|\nabla v^{m-1} + 2\,x\right|^2 dx$$

where $\mathcal{B}(x) = \mathsf{g}^{2p}(x) = \left(1+|x|^2\right)^{-\frac{1}{1-m}}$ (with appropriate normalizations)

Linearized entropy-entropy production inequality

(BBDGV)... Linearization: Let $v_\varepsilon = \mathcal{B}\left(1 + \varepsilon\,\mathcal{B}^{1-m}\,f\right)$

$$\underbrace{\mathcal{I}[v_\varepsilon]}_{\sim\,\varepsilon^2\int_{\mathbb{R}^d}|\nabla f|^2\,\mathcal{B}\,dx} \quad \ge \quad 4 \quad \underbrace{\mathcal{F}[v_\varepsilon]}_{\sim\,\varepsilon^2\int_{\mathbb{R}^d}|f|^2\,\mathcal{B}^{2-m}\,dx}$$

Hardy–Poincaré inequality: with $\mathcal{B}^{2-m} = \frac{\mathcal{B}}{1+|x|^2}$

$$\Lambda_{m,d}\int_{\mathbb{R}^d} f^2\,\mathcal{B}^{2-m}\,dx \le \int_{\mathbb{R}^d}|\nabla f|^2\,\mathcal{B}\,dx \quad \forall\ f\in \mathrm{H}^1(\mathcal{B}\,dx)\,, \quad \int_{\mathbb{R}^d} f\,\mathcal{B}^{2-m}\,dx = 0$$

- asymptotic decay rates = rates of the linearized FDE equation

$$0 = \partial_t v + \nabla\cdot\left(v\,\nabla\left(v^{m-1} - \mathcal{B}^{m-1}\right)\right)$$

$$\sim \varepsilon\,\mathcal{B}^{2-m}\left(\partial_t f - (1-m)\,\mathcal{B}^{m-2}\nabla\cdot\left(\mathcal{B}\,\nabla f\right)\right)$$

- same rate in the nonlinear regime (Bakry-Emery)
- much more (BDNS): stability results... but the difficulty lies in the justification of the Taylor expansion

The subcritical Keller-Segel model

(Campos, JD)
(Dávila, JD, del Pino, Musso, Wei)

The subcritical Keller-Segel model

$M = \int_{\mathbb{R}^2} n_0 \, dx \le 8\pi$: global existence (W. Jäger, S. Luckhaus 1992), (JD, B. Perthame 2004)

If u solves

$$\frac{\partial u}{\partial t} = \nabla \cdot [u \, (\nabla (\log u) - \nabla v)]$$

the *free energy*

$$F[u] := \int_{\mathbb{R}^2} u \log u \, dx - \frac{1}{2} \int_{\mathbb{R}^2} u \, v \, dx$$

satisfies

$$\frac{d}{dt} F[u(t,\cdot)] = - \int_{\mathbb{R}^2} u \left| \nabla (\log u) - \nabla v \right|^2 dx$$

The *logarithmic HLS inequality* (E. Carlen, M. Loss 1992)

F is bounded from below if and only if $M \le 8\pi$

Time-dependent rescaling

$$u(x,t) = \frac{1}{R^2(t)}\, n\left(\frac{x}{R(t)}, \tau(t)\right) \quad \text{and} \quad v(x,t) = c\left(\frac{x}{R(t)}, \tau(t)\right)$$

with $R(t) = \sqrt{1+2\,t}$ and $\tau(t) = \log R(t)$

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n\,(\nabla c - x)) & x \in \mathbb{R}^2\,,\ t > 0 \\ c = -\dfrac{1}{2\pi}\, \log|\cdot| * n & x \in \mathbb{R}^2\,,\ t > 0 \\ n(\cdot, t=0) = n_0 \ge 0 & x \in \mathbb{R}^2 \end{cases}$$

(A. Blanchet, JD, B. Perthame 2006)
The convergence in self-similar variables

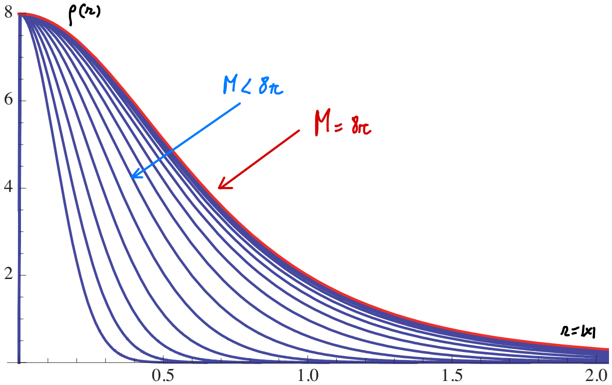
$$\lim_{t\to\infty} \|n(\cdot,\cdot+t) - n_\infty\|_{\mathrm{L}^1(\mathbb{R}^d)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \|\nabla c(\cdot,\cdot+t) - \nabla c_\infty\|_{\mathrm{L}^2(\mathbb{R}^d)} = 0$$

means *intermediate asymptotics* in original variables:

$$\|u(x,t) - \tfrac{1}{R^2(t)}\, n_\infty\left(\tfrac{x}{R(t)}, \tau(t)\right)\|_{L^1(\mathbb{R}^2)} \searrow 0$$

The stationary solution in self-similar variables

$$n_\infty = M\,\frac{e^{\,c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2}\,dx} = -\Delta c_\infty\,,\qquad c_\infty = -\frac{1}{2\pi}\log|\cdot| * n_\infty$$



Linearization

We can introduce two functions f and g such that

$$n = n_\infty (1 + f) \quad \text{and} \quad c = c_\infty (1 + g) = (-\Delta)^{-1} n$$

and rewrite the Keller-Segel model as

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f + \frac{1}{n_\infty} \nabla (f\, n_\infty \nabla (c_\infty\, g))$$

where the linearized operator is

$$\mathcal{L}\, f = \frac{1}{n_\infty} \nabla \cdot \big(n_\infty \nabla (f - c_\infty\, g) \big)$$

and

$$-\Delta (c_\infty\, g) = n_\infty\, f$$

Spectrum of $\mathcal{L}$ (lowest eigenvalues only)

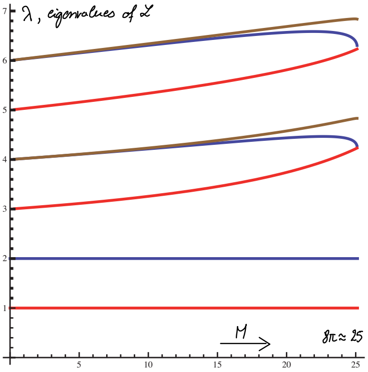


Figure: The lowest eigenvalues of $-\mathcal{L} = (-\Delta)^{-1}(n\,f)$

Functional setting...

Lemma (A. Blanchet, JD, B. Perthame)

Sub-critical HLS inequality (A. Blanchet, JD, B. Perthame)

$$F[n] := \int_{\mathbb{R}^2} n \log\left(\frac{n}{n_\infty}\right) dx - \frac{1}{2}\int_{\mathbb{R}^2} (n - n_\infty)\,(c - c_\infty)\, dx \geq 0$$

achieves its minimum for $n = n_\infty$

$$\mathsf{Q}_1[f] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2}\, F[n_\infty(1 + \varepsilon\, f)] \geq 0$$

if $\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0$. Notice that f_0 generates the kernel of Q_1

Lemma (J. Campos, JD)

Poincaré type inequality. For any $f \in H^1(\mathbb{R}^2, n_\infty\, dx)$ *such that* $\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0$, *we have*

$$\int_{\mathbb{R}^2} |\nabla(-\Delta)^{-1}(f\, n_\infty)|^2\, n_\infty\, dx = \int_{\mathbb{R}^2} |\nabla(g\, c_\infty)|^2\, n_\infty\, dx \leq \int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx$$

... and eigenvalues

With g such that $-\Delta(g\,c_\infty) = f\,n_\infty$, Q_1 determines a scalar product

$$\langle f_1, f_2\rangle := \int_{\mathbb{R}^2} f_1\,f_2\,n_\infty\,dx - \int_{\mathbb{R}^2} f_1\,n_\infty\,(g_2\,c_\infty)\,dx$$

on the orthogonal space to f_0 in $L^2(n_\infty\,dx)$

$$\mathsf{Q}_2[f] := \int_{\mathbb{R}^2} |\nabla(f - g\,c_\infty)|^2\,n_\infty\,dx \quad \text{with} \quad g = -\frac{1}{c_\infty}\,\frac{1}{2\pi}\,\log|\cdot| * (f\,n_\infty)$$

is a positive quadratic form, whose polar operator is the self-adjoint operator $\mathcal{L}$

$$\langle f, \mathcal{L}\,f\rangle = \mathsf{Q}_2[f] \quad \forall\ f \in \mathcal{D}(\mathsf{L}_2)$$

Lemma (J. Campos, JD)

$\mathcal{L}$ has pure discrete spectrum and its lowest eigenvalue is 1

A simple Cucker-Smale mean-field model

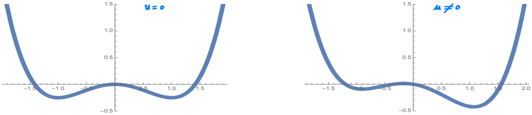
(Xingyu Li)

A simple version of the Cucker-Smale model

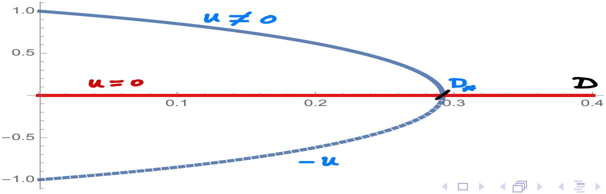
(J. Tugaut, 2014), (A. Barbaro, J. Cañizo, J.A. Carrillo, and P. Degond, 2016), (X. Li)

A model for bird flocking (simplified version)

$$\frac{\partial f}{\partial t} = D\,\Delta_v f + \nabla_v \cdot \left(\nabla_v \phi(v)\, f - \mathbf{u}_f\, f\right)$$



where $\mathbf{u}_f = \int v\, f\, dv$ is the average velocity and $\phi(v) = \frac{1}{4}|v|^4 - \frac{1}{2}|v|^2$



Relative entropy and related quantities

$$\frac{d}{dt}\mathcal{F}_{\mathbf{u}}[f(t,\cdot)] = -\mathcal{I}[f]$$

- *Relative entropy* with respect to a stationary solution $f_{\mathbf{u}}$

$$\mathcal{F}_{\mathbf{u}}[f] = D\int_{\mathbb{R}^d} f\,\log\left(\frac{f}{f_{\mathbf{u}}}\right)dv - \frac{1}{2}\,|\mathbf{u}_f - \mathbf{u}|^2$$

- *Relative Fisher information*

$$\mathcal{I}[f] := \int_{\mathbb{R}^d}\left|D\,\frac{\nabla f}{f} + \alpha\,v\,|v|^2 + (1-\alpha)\,v - \mathbf{u}_f\right|^2 f\,dv$$

- *Non-equilibrium Gibbs state*

$$G_f(v) := \frac{e^{-\frac{1}{D}\left(\frac{1}{2}\,|v-\mathbf{u}_f|^2+\frac{\alpha}{4}\,|v|^4-\frac{\alpha}{2}\,|v|^2\right)}}{\int_{\mathbb{R}^d} e^{-\frac{1}{D}\left(\frac{1}{2}\,|v-\mathbf{u}_f|^2+\frac{\alpha}{4}\,|v|^4-\frac{\alpha}{2}\,|v|^2\right)}\,dv}$$

Stability and coercivity

(X. Li)

$$Q_{1,\mathbf{u}}[g] := \lim_{\varepsilon\to0} \frac{2}{\varepsilon^2}\,\mathcal{F}\big[f_{\mathbf{u}}(1+\varepsilon\,g)\big] = D\int_{\mathbb{R}^d} g^2\,f_{\mathbf{u}}\,dv - D^2\,|\mathbf{v}_g|^2$$

where $\mathbf{v}_g := \frac{1}{D}\int_{\mathbb{R}^d} v\,g\,f_{\mathbf{u}}\,dv$

$$Q_{2,\mathbf{u}}[g] := \lim_{\varepsilon\to0} \frac{1}{\varepsilon^2}\,\mathcal{I}\big[f_{\mathbf{u}}\,(1+\varepsilon\,g)\big] = D^2\int_{\mathbb{R}^d} \left|\nabla g - \mathbf{v}_g\right|^2\,f_{\mathbf{u}}\,dv$$

Stability: $Q_{1,\mathbf{u}} \ge 0$ *?*

Coercivity: $Q_{2,\mathbf{u}} \ge \lambda\,Q_{1,\mathbf{u}}$ *for some* $\lambda > 0$ *?*

$$Q_{2,\mathbf{u}}[g] \ge \mathcal{C}_D\,\big(1-\kappa(D)\big)\,\frac{(\mathbf{v}_g\cdot\mathbf{u})^2}{|\mathbf{v}_g|^2\,|\mathbf{u}|^2}\,Q_{1,\mathbf{u}}[g]$$

$\kappa(D) < 1$ and as a special case, if $\mathbf{u} = \mathbf{u}[f]$, then

$$Q_{2,\mathbf{u}}[g] \ge \mathcal{C}_D\,\big(1-\kappa(D)\big)\,Q_{1,\mathbf{u}}[g]$$

An exponential rate of convergence for partially symmetric solutions in the polarized case

Proposition (X. Li)

Let $\alpha > 0$, $D > 0$ and consider a solution $f \in C^0\big(\mathbb{R}^+, \mathrm{L}^1(\mathbb{R}^d)\big)$ with initial datum $f_{\mathrm{in}} \in \mathrm{L}^1_+(\mathbb{R}^d)$ such that $\mathcal{F}[f_{\mathrm{in}}] < \mathcal{F}[f_{\mathbf{0}}]$ and $\mathbf{u}_{f_{\mathrm{in}}} = (u, 0 \ldots 0)$ for some $u \neq 0$. We further assume that $f_{\mathrm{in}}(v_1, v_2, \ldots v_{i-1}, v_i, \ldots) = f_{\mathrm{in}}(v_1, v_2, \ldots v_{i-1}, -v_i, \ldots)$ for any $i = 2, 3, \ldots d$. Then

$$0 \le \mathcal{F}[f(t,\cdot)] - \mathcal{F}[f_{\mathbf{u}}] \le C\, e^{-\lambda\, t} \quad \forall\, t \ge 0$$

holds with $\lambda = \mathcal{C}_D\,\big(1 - \kappa(D)\big) > 0$

Hypocoercivity methods
an overview

- (JD, Mouhot, Schmeiser, 2015)
- (Bouin, JD, Mischler, Mouhot, Schmeiser, 2020) Hypocoercivity without confinement
- (Arnold, JD, Schmeiser, Wöhrer) Sharpening of decay rates in Fourier based hypocoercivity methods

H^1-hypocoercivity: an example

$$\frac{\partial f}{\partial t} + \mathsf{T} f = \Delta_v f + \nabla_v \cdot (v\, f)\;, \quad \mathsf{T} f := v \cdot \nabla_x f - x \cdot \nabla_v f$$

(JD, X. Li) take $h = (f/f_*)^{2/p}$, $p \in (1,2)$

$$\frac{\partial h}{\partial t} + \mathsf{T} h = \mathsf{L}\, h + \frac{2-p}{p}\,\frac{|\nabla_v h|^2}{h}\,, \qquad \mathsf{L}\, h := \Delta_v h - v \cdot \nabla_v h$$

Twisted Fisher information

$$\mathcal{J}_\lambda[h] = (1-\lambda) \int_{\mathbb{R}^d} |\nabla_v h|^2\, d\mu + (1-\lambda) \int_{\mathbb{R}^d} |\nabla_x h|^2\, d\mu + \lambda \int_{\mathbb{R}^d} |\nabla_x h + \nabla_v h|^2\, d\mu$$

Theorem (JD, Li)

For an appropriate choice of $t \mapsto \lambda(t)$, there is a function $t \mapsto \rho(t) > 1$ a.e.

$$\frac{d}{dt} \mathcal{J}_{\lambda(t)}[h(t,\cdot)] \le -\rho(t)\, \mathcal{J}_{\lambda(t)}[h(t,\cdot)] \quad \forall\, t \ge 0$$

and $\mathcal{J}_{\lambda(t)}[h(t,\cdot)] \le \mathcal{J}_{1/2}[h_0]\, \exp\left(-\int_0^t \rho(s)\, ds\right)$

L^2-hypocoercivity: the strategy

(JD, Mouhot, Schmeiser) Π is the orthogonal projection on $\mathrm{Ker}(\mathsf{L})$

$$\varepsilon\,\frac{dF}{dt} + \mathsf{T}F = \frac{1}{\varepsilon}\,\mathsf{L}F$$

$F_\varepsilon = F_0 + \varepsilon\,F_1 + \varepsilon^2\,F_2 + \mathcal{O}(\varepsilon^3)$ as $\varepsilon \to 0_+$, $u = F_0 = \Pi F_0$

$$\partial_t u + (\mathsf{T}\Pi)^*\,(\mathsf{T}\Pi)\,u = 0$$

▹ Main assumption: *macroscopic coercivity* (Poincaré inequality)

$$\|\mathsf{T}\Pi F\|^2 \ge \lambda_M\,\|\Pi F\|^2$$

$\varepsilon = 1$: the estimate $\frac{1}{2}\,\frac{d}{dt}\|F\|^2 = \langle \mathsf{L}F, F\rangle \le -\lambda_m\,\|(1-\Pi)F\|^2$ is not enough to conclude that $\|F(t,\cdot)\|^2$ decays exponentially

The operator $\mathsf{A} := \big(1 + (\mathsf{T}\Pi)^*\mathsf{T}\Pi\big)^{-1}(\mathsf{T}\Pi)^*$ is such that

$$\langle \mathsf{A}\mathsf{T}\Pi F, F\rangle \ge \frac{\lambda_M}{1+\lambda_M}\,\|\Pi F\|^2$$

and we can use the L^2 entropy / Lyapunov functional

$$\mathsf{H}[F] := \frac{1}{2}\,\|F\|^2 + \delta\,\mathrm{Re}\langle \mathsf{A}F, F\rangle$$

H^{-1}-hypocoercivity

▷ (S. Armstrong and J.-C. Mourrat). Variational methods for the kinetic Fokker-Planck equation. arXiv:1902.04037, 2019

▷ (G. Brigati). Time averages for kinetic Fokker-Planck equations

Consider the *kinetic-Ornstein-Uhlenbeck equation*

$$\partial_t h + v \cdot \nabla_x h = \Delta_\alpha h := \Delta_v h - \alpha\, v\, \langle v \rangle^{\alpha-2} \cdot \nabla_v h\,, \quad h(0,\cdot,\cdot) = h_0$$

on $\mathbb{R}^+ \times (0,L)^d \times \mathbb{R}^d$ (periodic boundary conditions in x) with local equilibrium $\gamma_\alpha(v) = Z_\alpha^{-1}\, e^{-\langle v \rangle^\alpha}$

Theorem (Brigati)

Let $\alpha \ge 1$, $L > 0$ and $\tau > 0$. There exists a constant $\lambda > 0$ such that, for all $h_0 \in \mathrm{L}^2(dx\, d\gamma_\alpha)$ with zero-average,

$$\int_t^{t+\tau} \|h(s,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\, d\gamma_\alpha)}\, ds \le \|h_0\|^2_{\mathrm{L}^2(dx\, d\gamma_\alpha)}\, e^{-\lambda\, t} \quad \forall\, t \ge 0$$

H^{-1}-hypocoercivity

$\Omega = (0,\tau) \times (0,L)^d$, and $\alpha > 0$

▷ *Averaging lemma*

$$\|\nabla_{t,x}\rho_h\|^2_{\mathrm{H}^{-1}(\Omega)} \le d_\alpha \left(\|h-\rho_h\|^2_{\mathrm{L}^2(dt\,dx\,d\gamma_\alpha)} + \|\partial_t h + v\cdot\nabla_x h\|^2_{\mathrm{L}^2(\Omega;H^{-1}_\alpha)} \right)$$

▷ *A generalized Poincaré inequality* (based on JL Lions' lemma)

$$\|h\|^2_{\mathrm{L}^2(dt\,dx\,d\gamma_\alpha)} \le C \left(\|h-\rho_h\|^2_{\mathrm{L}^2(dt\,dx\,d\gamma_\alpha)} + \|\partial_t h + v\cdot\nabla_x h\|^2_{\mathrm{L}^2(\Omega;H^{-1}_\alpha)} \right)$$

With several *conservation laws*

(Carrapatoso, JD, Hérau, Mischler, Mouhot). Weighted Korn and Poincaré-Korn inequalities in the Euclidean space and associated operators. https://hal.archives-ouvertes.fr/hal-03059166

(Carrapatoso, JD, Hérau, Mischler, Mouhot, Schmeiser). Special modes and hypocoercivity for linear kinetic equations with several conservation laws and a confining potential https://hal.archives-ouvertes.fr/hal-03222748

$$\frac{\partial f}{\partial t} + \mathsf{T} f = \mathsf{L} f\,, \quad \mathsf{T} f := v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f$$

Collision invariants $\int_{\mathbb{R}^d} \left(1, v, |v|^2\right) \mathsf{L} f \, dv = 0$

Difficulties:
– Rigid motions
– Time periodic solutions (breathers, rotations in phase space) when ϕ is quadratic

The Vlasov-Poisson-Fokker-Planck system

▷ Linearized Vlasov-Poisson-Fokker-Planck system

▷ A result in the non-linear case, $d = 1$

- (Addala, JD, Li, Tayeb) L^2-Hypocoercivity and large time asymptotics of the linearized Vlasov-Poisson-Fokker-Planck system. Preprint hal-02299535 and arxiv: 1909.12762

- (Hérau, Thomann, 2016), (Herda, Rodrigues, 2018)

Linearized Vlasov-Poisson-Fokker-Planck system

In collaboration with Lanoir Addala, Xingyu Li and Lazhar M. Tayeb

Linearized Vlasov-Poisson-Fokker-Planck system

The *Vlasov-Poisson-Fokker-Planck system* in presence of an external potential V is

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f - (\nabla_x V + \nabla_x \phi) \cdot \nabla_v f &= \Delta_v f + \nabla_v \cdot (v\, f) \\ -\Delta_x \phi = \rho_f &= \int_{\mathbb{R}^d} f\, dv \end{aligned} \tag{VPFP}$$

Linearized problem around $f_\star$: $f = f_\star\,(1 + \eta\, h)$, $\iint_{\mathbb{R}^d \times \mathbb{R}^d} h\, f_\star\, dx\, dv = 0$

$$\partial_t h + v \cdot \nabla_x h - (\nabla_x V + \nabla_x \phi_\star) \cdot \nabla_v h + v \cdot \nabla_x \psi_h - \Delta_v h + v \cdot \nabla_v h = \eta\, \nabla_x \psi_h \cdot \nabla_v h$$

$$-\Delta_x \psi_h = \int_{\mathbb{R}^d} h\, f_\star\, dv$$

Drop the $\mathcal{O}(\eta)$ term : *linearized Vlasov-Poisson-Fokker-Planck* / *Ornstein-Uhlenbeck* system

$$\begin{aligned} \partial_t h + v \cdot \nabla_x h - (\nabla_x V + \nabla_x \phi_\star) \cdot \nabla_v h + v \cdot \nabla_x \psi_h - \Delta_v h + v \cdot \nabla_v h &= 0 \\ -\Delta_x \psi_h = \int_{\mathbb{R}^d} h\, f_\star\, dv\,, \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} h\, f_\star\, dx\, dv &= 0 \end{aligned} \tag{VPFPlin}$$

Hypocoercivity

Let us define the norm

$$\|h\|^2 := \iint_{\mathbb{R}^d\times\mathbb{R}^d} h^2\, f_\star\, dx\, dv + \int_{\mathbb{R}^d} |\nabla_x \psi_h|^2\, dx$$

Theorem

Let us assume that $d \ge 1$, $V(x) = |x|^\alpha$ for some $\alpha > 1$ and $M > 0$. Then there exist two positive constants $\mathcal{C}$ and λ such that any solution h of (VPFPlin) with an initial datum h_0 of zero average with $\|h_0\|^2 < \infty$ is such that

$$\|h(t,\cdot,\cdot)\|^2 \le \mathcal{C}\, \|h_0\|^2\, e^{-\lambda t} \quad \forall\, t \ge 0$$

Diffusion limit

Linearized problem in the parabolic scaling

$$\varepsilon\,\partial_t h + v\cdot\nabla_x h - (\nabla_x V + \nabla_x\phi_\star)\cdot\nabla_v h + v\cdot\nabla_x\psi_h - \frac{1}{\varepsilon}\big(\Delta_v h - v\cdot\nabla_v h\big) = 0$$
$$-\Delta_x\psi_h = \int_{\mathbb{R}^d} h\,f_\star\,dv\,,\quad \iint_{\mathbb{R}^d\times\mathbb{R}^d} h\,f_\star\,dx\,dv = 0 \tag{VPFPscal}$$

Expand $h_\varepsilon = h_0 + \varepsilon\,h_1 + \varepsilon^2\,h_2 + \mathcal{O}(\varepsilon^3)$ as $\varepsilon\to 0_+$. With $W_\star = V + \phi_\star$

$$\begin{aligned}
\varepsilon^{-1}: &\qquad \Delta_v h_0 - v\cdot\nabla_v h_0 = 0\\
\varepsilon^{0}: &\qquad v\cdot\nabla_x h_0 - \nabla_x W_\star\cdot\nabla_v h_0 + v\cdot\nabla_x\psi_{h_0} = \Delta_v h_1 - v\cdot\nabla_v h_1\\
\varepsilon^{1}: &\qquad \partial_t h_0 + v\cdot\nabla_x h_1 - \nabla_x W_\star\cdot\nabla_v h_1 = \Delta_v h_2 - v\cdot\nabla_v h_2
\end{aligned}$$

With $u = \Pi h_0$, $-\Delta\psi = u\,\rho_\star$, $w = u + \psi$,

$$u = h_0\,,\quad v\cdot\nabla_x w = \Delta_v h_1 - v\cdot\nabla_v h_1$$

from which we deduce that $h_1 = -\,v\cdot\nabla_x w$ and

$$\partial_t u - \Delta w + \nabla_x W_\star\cdot\nabla u = 0$$

Rates of convergence

Results in the diffusion limit / in the non-linear case

Theorem

Let us assume that $d \geq 1$, $V(x) = |x|^\alpha$ for some $\alpha > 1$ and $M > 0$. For any $\varepsilon > 0$ small enough, there exist two positive constants $\mathcal{C}$ and λ, which do not depend on ε, such that any solution h of (VPFPscal) *with an initial datum h_0 of zero average satisfies*

$$\|h(t,\cdot,\cdot)\|^2 \leq \mathcal{C}\,\|h_0\|^2\,e^{-\lambda t} \quad \forall\, t \geq 0$$

Corollary

Assume that $d = 1$, $V(x) = |x|^\alpha$ for some $\alpha > 1$ and $M > 0$. If f solves (VPFP) *with initial datum $f_0 = (1 + h_0)\,f_\star$ such that h_0 has zero average, $\|h_0\|^2 < \infty$ and $(1 + h_0) \geq 0$, then*

$$\|h(t,\cdot,\cdot)\|^2 \leq \mathcal{C}\,\|h_0\|^2\,e^{-\lambda t} \quad \forall\, t \geq 0$$

holds with $h = f/f_\star - 1$ for some positive constants $\mathcal{C}$ and λ

These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Lectures/
▷ Lectures

The papers can be found at

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For final versions, use `Dolbeault` as login and `Jean` as password

Thank you for your attention !

Decay and convergence rates for kinetic equations

L^2 hypocoercivity: what can we do when at least one of the coercivity conditions (microscopic coercivity or macroscopic coercivity) is missing ?

In collaboration with Emeric Bouin, Stéphane Mischler, Clément Mouhot, Christian Schmeiser and Laurent Lafleche

The global picture: from diffusive to kinetic

- Depending on the local equilibria and on the external potential (which are Poincaré type inequalities) can be replaced by other functional inequalities:

▷ *microscopic coercivity*

$$-\langle \mathsf{L} F, F\rangle \geq \lambda_m \,\|(1-\Pi)F\|^2$$

$\Longrightarrow$ ***weak Poincaré inequalities*** or ***Hardy-Poincaré inequalities***

▷ *macroscopic coercivity*

$$\|\mathsf{T}\Pi F\|^2 \geq \lambda_M \,\|\Pi F\|^2$$

$\Longrightarrow$ ***Nash inequality***, ***weighted Nash*** or ***Caffarelli-Kohn-Nirenberg inequalities***

- This can be done at the level of the *diffusion equation* (homogeneous case) or at the level of the *kinetic equation* (non-homogeneous case)

Diffusion (Fokker-Planck) equations

Potential	$V = 0$	$V(x) = \gamma \log \lvert x \rvert$ $\gamma < d$	$V(x) = \lvert x \rvert^\alpha$ $\alpha \in (0,1)$	$V(x) = \lvert x \rvert^\alpha$ $\alpha \geq 1$
Inequality	Nash	Caffarelli-Kohn -Nirenberg	Weak Poincaré or Weighted Poincaré	Poincaré
Asymptotic behavior	$t^{-d/2}$ decay	$t^{-(d-\gamma)/2}$ decay	$t^{-\mu}$ or $t^{-\frac{k}{2(1-\alpha)}}$ convergence	$e^{-\lambda t}$ convergence

Table 1: $\partial_t u = \Delta u + \nabla \cdot (u \nabla V)$

Kinetic Fokker-Planck equations

B = Bouin, L = Lafleche, M = Mouhot, MM = Mischler, Mouhot
S = Schmeiser

Potential	$V=0$	$V(x)=\gamma\,\log\lvert x\rvert$ $\gamma<d$	$V(x)=\lvert x\rvert^\alpha$ $\alpha\in(0,1)$	$V(x)=\lvert x\rvert^\alpha$ $\alpha\ge1$, or $\mathbb{T}^d$ Macro Poincaré
Micro Poincaré $F(v)=e^{-\langle v\rangle^\beta}$, $\beta\ge1$	BDMMS: $t^{-d/2}$ decay	BDS: $t^{-(d-\gamma)/2}$ decay	Cao: e^{-t^b}, $b<1$, $\beta=2$ convergence	DMS, Mischler- Mouhot $e^{-\lambda t}$ convergence
$F(v)=e^{-\langle v\rangle^\beta}$, $\beta\in(0,1)$	BDLS: $t^{-\zeta}$, $\zeta=$ $\min\left\{\frac{d}{2},\frac{k}{\beta}\right\}$ decay			
$F(v)=\langle v\rangle^{-d-\beta}$	BDLS, fractional $t^{-\zeta}$, $\zeta=$ $\min\left\{\frac{k}{2},3\,d\,\zeta_0\right\}$ $\zeta_0=$ $\max\{6,\beta+2\}$			

Table 1: $\partial_t f+v\cdot\nabla_x f=F\,\nabla_v\big(F^{-1}\,\nabla_v f\big)$. Notation: $\langle v\rangle=\sqrt{1+|v|^2}$