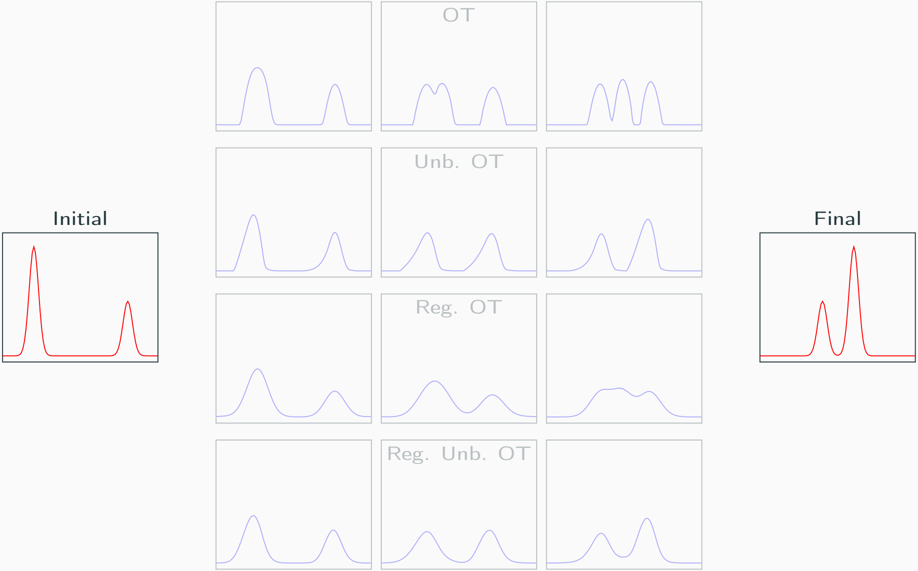
From branching Brownian motion to regularized unbalanced optimal transport

Aymeric Baradat

23 juin 2021

Work **in progress** in collaboration with **Hugo Lavenant**, from **Bocconi University** in Milan.

Different models of optimal transport



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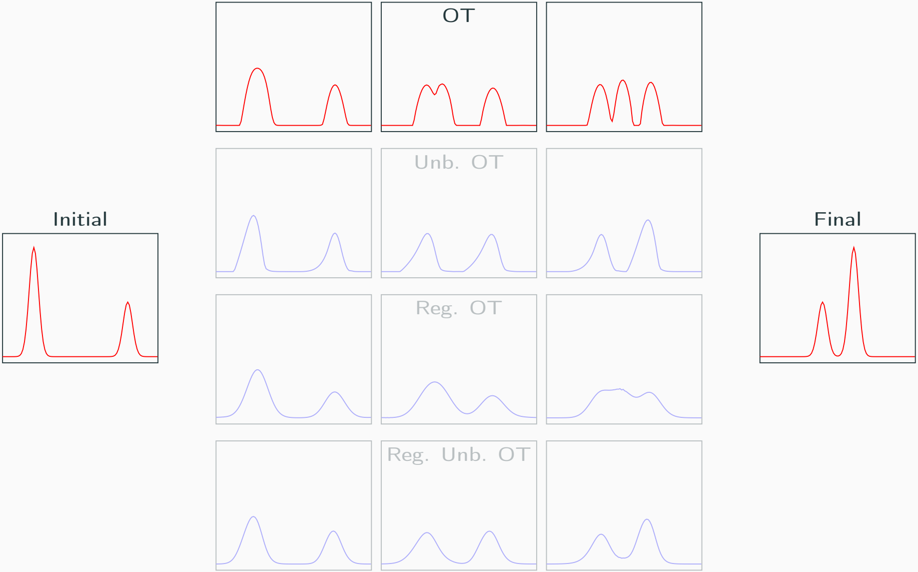

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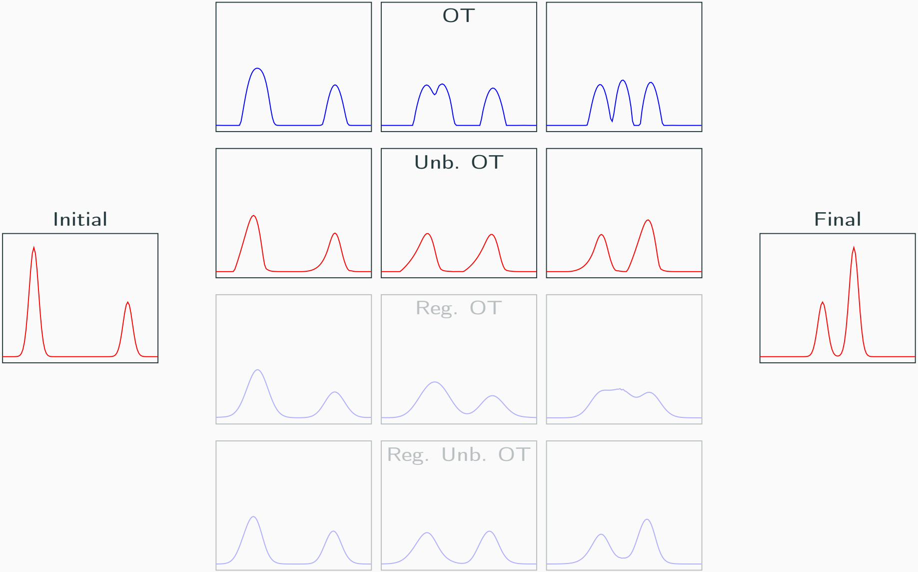

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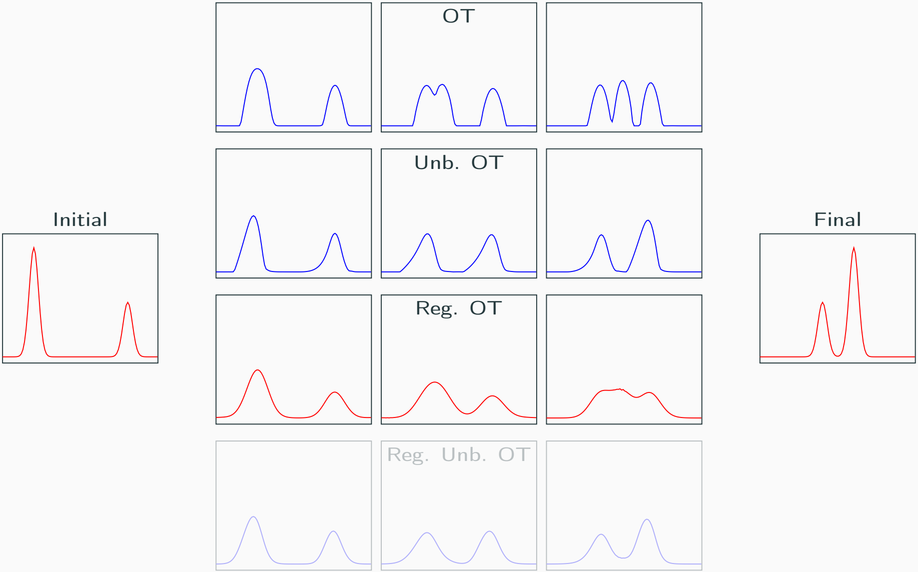

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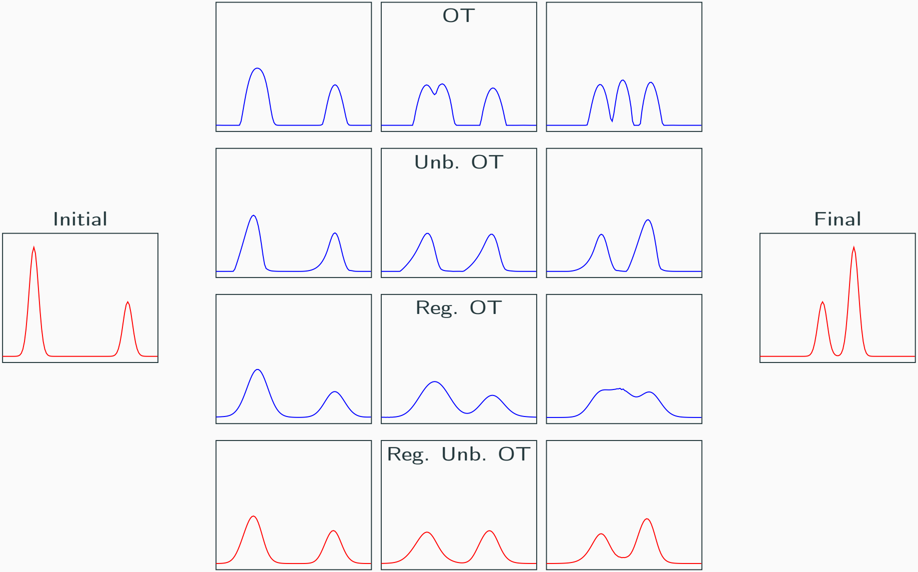

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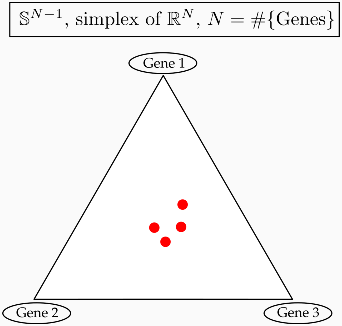

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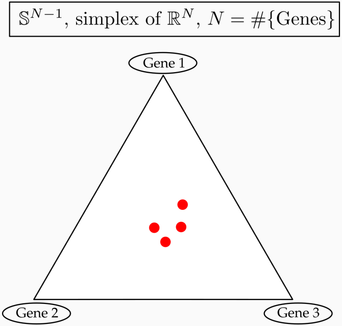

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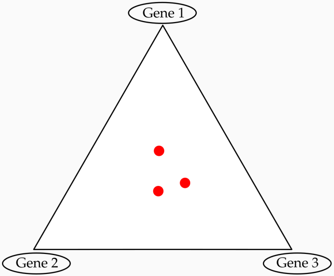

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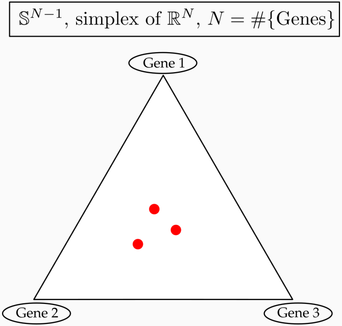

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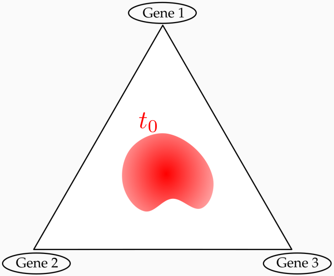

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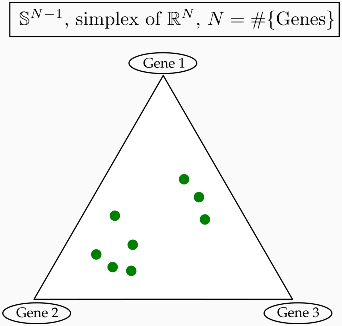

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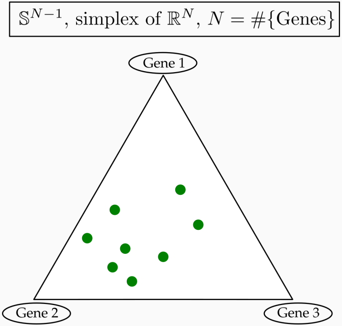

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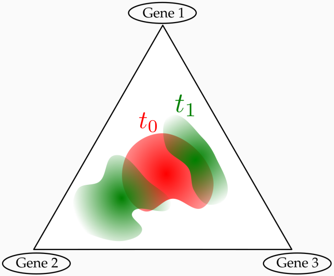

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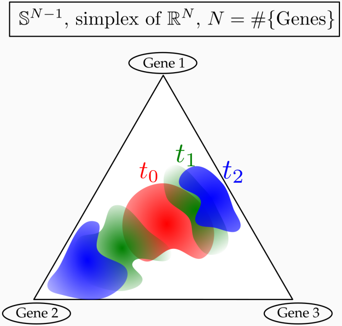

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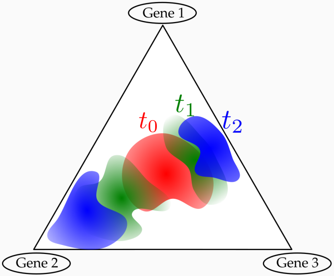

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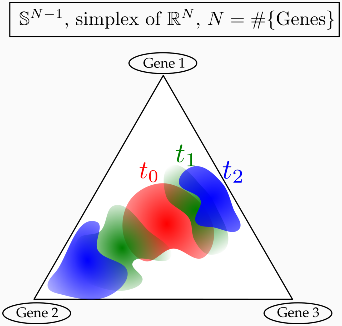

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Regularized unbalanced optimal transport (RUOT)

Let $\rho_0, \rho_1 \in \mathcal{M}_+(\mathbb{T}^d)$. Our optimization problem reads

Among those fields $\rho = \rho(t,x)$, $v = v(t,x)$, $r = r(t,x)$ satisfying:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = \dfrac{\nu}{2}\Delta\rho + \rho r, \\ \rho|_{t=0} = \rho_0, \quad \rho|_{t=1} = \rho_1, \end{cases}$$

find the ones minimizing:

$$\int_0^1 \int \left\{ \frac{1}{2}|v(t,x)|^2 + \Psi(r(t,x)) \right\} \rho(t, \mathrm{d}x)\, \mathrm{d}x\, \mathrm{d}t.$$

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Question: How to chose Ψ?

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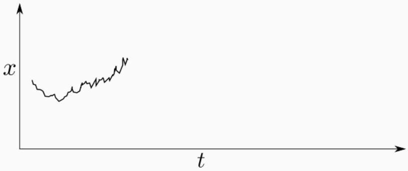

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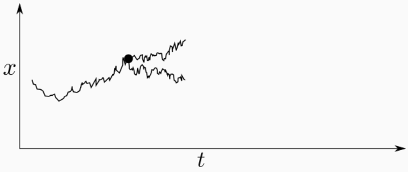

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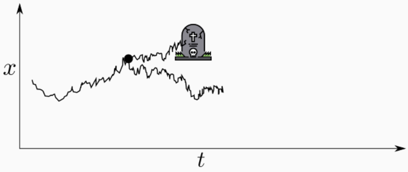

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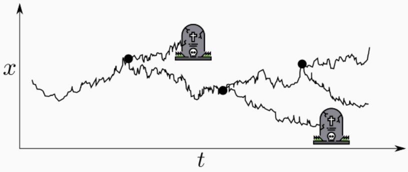

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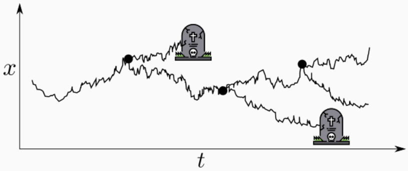
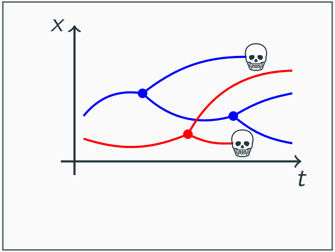


Remark: λ and $\boldsymbol{p}$ could depend on t and x: $\lambda(t, x)\,\mathrm{d}t$ represents the probability for a particle in (t, x) to die between t and $t + \mathrm{d}t$. Then, $p_k(t, x)$ is the probability that it is replaced by k particles.

Entropic minimization w.r.t. BBM



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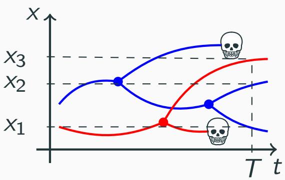

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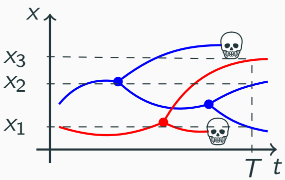

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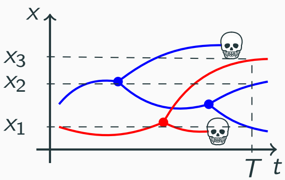

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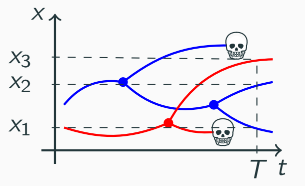

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- $R \in \mathcal{P}(\Omega)$: law of BBM of parameters ν, λ and $\boldsymbol{p} = (p_k)_k$, of initial law $R_0 \in \mathcal{P}(\mathcal{M}_\delta(\mathbb{T}^d))$.

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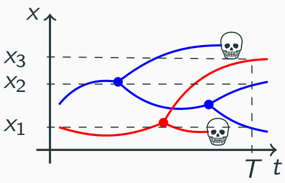

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Fixing the marginals $\rho_0, \rho_1 \in \mathcal{M}_+(\mathbb{T}^d)$, our minimization problem reads:

Among those $P \in \mathcal{P}(\Omega)$ satisfying for all $A \in \mathcal{B}(\mathbb{T}^d)$:

$$\mathbb{E}_P[\omega_0(A)] = \rho_0(A) \qquad \text{and} \qquad \mathbb{E}_P[\omega_1(A)] = \rho_1(A).$$

find the ones minimizing

$$H(P|R) := \mathbb{E}_R\left[\frac{\mathrm{d}P}{\mathrm{d}R}\log\frac{\mathrm{d}P}{\mathrm{d}R}\right] = \mathbb{E}_P\left[\log\frac{\mathrm{d}P}{\mathrm{d}R}\right].$$

Equivalence of the models

RUOT

Let $\nu > 0$ and Ψ a function.

Find (ρ, v, r) s.t.

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = \dfrac{\nu}{2}\Delta\rho + \rho r, \\ \rho|_{t=0} = \rho_0, \quad \rho|_{t=1} = \rho_1, \end{cases}$$

minimizing

$$\int_0^1 \int \left\{ \frac{|v|^2}{2} + \Psi(r) \right\} \rho \, \mathrm{d}x \, \mathrm{d}t.$$

Branching Schrödinger

R BBM of initial law R_0, of parameter ν, λ and $\boldsymbol{p}$.

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Main result (BL): If Ψ is defined *via* its Legendre transform by:

$$\Psi^*(s) = \nu\lambda \sum p_k \left\{ e^{(k-1)\frac{s}{\nu}} - 1 \right\},$$

The minimum in RUOT is the l.s.c. relaxation of the infimum in Branching Schrödinger, up to an initial term.

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Observation: Provided Ψ is defined as before,

$$\Psi(r) = \inf \left\{ h(\tilde{\boldsymbol{q}}|\boldsymbol{q}) \ : \ \sum (k-1) \tilde{q}_k = r \right\}.$$

A few ideas

Changing the law: Let us call $\boldsymbol{q} = \lambda \boldsymbol{p}$. Let P be the law of the BBM of drift $v = v(t,x)$, whose branching events are describe by $\tilde{\boldsymbol{q}} = \tilde{\boldsymbol{q}}(t,x)$, and of initial law P_0, then calling $\rho(t) := \mathbb{E}_P[\omega_t]$ and $r(t,x) := \sum_k (k-1) q_k$, we have:

$$\partial_t \rho + \operatorname{div}(\rho v) = \frac{\nu}{2} \Delta \rho + \rho r,$$

$$\nu H(P|R) = \nu H(P_0|R_0) + \int_0^1 \int \left\{ \frac{1}{2} |v(t,x)|^2 + h(\tilde{\boldsymbol{q}}(t,x)|\boldsymbol{q}) \right\} \rho(t, \mathrm{d}x) \, \mathrm{d}x \, \mathrm{d}t,$$

$$\text{where} \qquad h(\tilde{\boldsymbol{q}}|\boldsymbol{q}) := \sum_k \tilde{q}_k \log \frac{\tilde{q}_k}{q_k} + q_k - \tilde{q}_k.$$

Observation: Provided Ψ is defined as before,

$$\Psi(r) = \inf \left\{ h(\tilde{\boldsymbol{q}}|\boldsymbol{q}) \ : \ \sum (k-1) \tilde{q}_k = r \right\}.$$

A counter-example to equality: In RUOT, we can connect 0 to $\rho_1 \neq 0$. In branching Schrödinger it is not possible.

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